Some Extensions of Sum and Product Theorems on Relative order and Relative Lower Order of Entire Functions

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Abstract

Some basic properties of relative order and relative lower order of entire functions have been discussed in this paper.

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1 Introduction, Definitions and Notations

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding
to $f$ is defined as follows:

$$M_f(r) = \max |z| = r |f(z)|.$$  

If $f$ is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_f^{-1}(s) = \infty$. On the other hand, the Nevanlinna’s Characteristic function of $f$ denoted by $T_f(r)$ is defined as

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta$$

where

$$\log^+ x = \max (\log x, 0) \text{ for all } x \geq 0.$$  

For any two given entire functions $f$ and $g$, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of $f$ with respect to $g$ in terms of their maximum moduli. The order of an entire function $f$ which is generally used in computational purpose is defined in terms of the growth of $f$ with respect to the exponential function as

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$  

L. Bernal [1, 2] introduced the definition of relative order of an entire function $g$ with respect to an entire function $f$ denoted by $\rho_f(g)$ to avoid comparing growth just with $\exp z$ which is as follows:

$$\rho_f(g) = \inf \{ \mu > 0 : M_g(r) < M_f(r^\mu) \text{ for all } r > r_0 (\mu) > 0 \}$$

$$= \limsup_{r \to \infty} \frac{\log M_{f^{-1}} M_g(r)}{\log r}.$$  

Definitely the above definition coincides with the classical one [10] if $f(z) = \exp z$.

Similarly, one can define the relative lower order of $g$ with respect to $f$, denoted by $\lambda_f(g)$ as follows:

$$\lambda_f(g) = \liminf_{r \to \infty} \frac{\log M_{f^{-1}} M_g(r)}{\log r}.$$  

An entire function $g$ is said to be of regular relative growth with respect to $f$ if its relative order with respect to $f$ coincides with its relative lower order with respect to $f$. 

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During the past decades, several authors (see [5],[6],[7]) made close investigations on the properties of relative order of entire functions. In this connection the following definition is relevant:

A non-constant entire function \( f \) is said have the Property (A) if for any \( \sigma > 1 \) and for all large \( r \), \[ M_f(r)^2 \leq M_f(r^\sigma) \] holds. For examples of functions with or without the Property (A), one may see [2].

In this paper we wish to investigate some basic properties of relative order and relative lower order of entire functions under somewhat different conditions. We do not explain the standard definitions and notations of the theory of entire functions as those are available in [11].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Suppose \( f \) be an entire function and \( \alpha, \beta \) are such that \( \alpha > 1 \) and \( 0 < \beta < \alpha \). Then
\[ M_f(\alpha r) > \beta M_f(r). \]

Let \( f \) be an entire function satisfying the Property (A). Then for any positive integer \( n \) and for all sufficiently large \( r \),
\[ [M_f(r)]^n \leq M_f(r^\delta) \]
holds where \( \delta > 1 \).

Every entire function \( f \) satisfying the Property (A) is transcendental.

Let \( f \) be an entire function. Then for all sufficiently large values of \( r \),
\[ T_f(r) \leq \log M_f(r) \leq 3T_f(2r) \quad \text{cf. [9], p. 18} \]

3 Main Results

In this section we present the main results of the paper. First we recall related four theorems which are needed in order to prove our results.

Theorem A. [2] Let \( f_1, g_1 \) and \( g_2 \) be any three entire functions. Then
\[ \rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_{1}) \]
where \( \rho_{f_1}(g_{1}) = \max \{ \rho_{f_1}(g_k) \mid k = i = 1, 2 \} \). The sign of equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_1}(g_2) \).

Theorem B. [2,9] Let \( f_1, g_1 \) and \( g_2 \) be any three entire functions. Then
\[ \rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_1) \]
where \( \rho_{f_i} (g_i) = \max \{ \rho_{f_i} (g_k) \mid k = i, 1, 2 \} \). The sign of equality holds when \( \rho_{f_i} (g_1) \neq \rho_{f_i} (g_2) \). Similar results hold for the quotient \( \frac{g_1}{g_2} \) provided \( \frac{g_1}{g_2} \) is entire.

**Theorem C.** [3] Let \( f_1, f_2 \) and \( g_1 \) be any three entire functions. Then

\[
\lambda_{f_1+f_2} (g_1) \geq \lambda_{f_i} (g_1)
\]

where \( \lambda_{f_i} (g_1) = \min \{ \lambda_{f_k} (g_1) \mid k = i, 1, 2 \} \). The sign of equality holds when \( \lambda_{f_i} (g_1) \neq \lambda_{f_2} (g_1) \).

**Theorem D.** [3] Let \( f_1, f_2 \) and \( g_1 \) be any three entire functions. Then

\[
\lambda_{f_1-f_2} (g_1) \geq \lambda_{f_i} (g_1)
\]

where \( \lambda_{f_i} (g_1) = \min \{ \lambda_{f_k} (g_1) \mid k = i, 1, 2 \} \). The sign of equality holds when \( \lambda_{f_i} (g_1) \neq \lambda_{f_2} (g_1) \). Similar results hold for the quotient \( \frac{f_1}{f_2} \) provided \( \frac{f_1}{f_2} \) is entire.

Now we prove the following results of the paper:

Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions. Then

(i)

\[
\rho_{f_1+f_2} (g_1) \geq \rho_{f_i} (g_1)
\]

where \( \rho_{f_i} (g_1) = \min \{ \rho_{f_k} (g_1) \mid k = i, 1, 2 \} \) and \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \). The sign of equality holds when \( \rho_{f_i} (g_1) \neq \rho_{f_2} (g_1) \); and

(ii)

\[
\lambda_{f_i} (g_1 \pm g_2) \leq \lambda_{f_i} (g_i)
\]

where \( \lambda_{f_i} (g_i) = \max \{ \lambda_{f_j} (g_k) \mid k = i, 1, 2 \} \) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_i \). The sign of equality holds when \( \lambda_{f_i} (g_1) \neq \lambda_{f_i} (g_2) \).

From the definition of relative order and relative lower order of entire functions, we have for all sufficiently large values of \( r \) that

\[
M_{g_k} (r) \leq M_{f_k} \left( r^{(}\rho_{f_k} (g_k) + \varepsilon)\right),
\]

(1)

\[
M_{g_k} (r) \geq M_{f_k} \left( r^{(}\lambda_{f_k} (g_k) - \varepsilon)\right),
\]

\[
i.e., \ M_{f_k} (r) \leq M_{g_k} \left( r^{\frac{1}{\lambda_{f_k} (g_k) - \varepsilon}}\right),
\]

(2)

and also for a sequence values of \( r \) tending to infinity we get that

\[
M_{g_k} (r) \geq M_{f_k} \left( r^{(}\rho_{f_k} (g_k) - \varepsilon)\right),
\]

\[
i.e., \ M_{f_k} (r) \leq M_{g_k} \left( r^{\frac{1}{\rho_{f_k} (g_k) - \varepsilon}}\right),
\]

(3)
where $\varepsilon > 0$ is any arbitrary positive number and $i = 1, 2$.

**Case I.** If $\rho_{f_1 \pm f_2} (g_1) = \infty$ then $\rho_{f_1 \pm f_2} (g_1) \geq \rho_{f_1} (g_1)$ is obvious. So we suppose that $\rho_{f_1 \pm f_2} (g_1) < \infty$. We can clearly assume that $\rho_{f_1} (g_1) \mid i = 1, 2$ is finite. Also suppose that $\rho_{f_i} (g_1) \leq \rho_{f_k} (g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$ and $g_1$ is of regular relative growth with respect to at least any one of $f_1$ or $f_2$. Now in view of (2), (3) and Lemma 2, we obtain for a sequence of values of $r$ tending to infinity that

$$M_{f_1 \pm f_2} (r) < M_{f_1} (r) + M_{f_2} (r)$$
i.e., $$M_{f_1 \pm f_2} (r) < \sum_{k=1}^{2} M_{g_1} \left( \frac{1}{r^{(\rho_{g_k} (g_1) - \varepsilon)}} \right)$$
i.e., $$M_{f_1 \pm f_2} (r) < 2M_{g_1} \left( \frac{1}{r^{(\rho_{g_1} (g_1) - \varepsilon)}} \right)$$
i.e., $$M_{f_1 \pm f_2} \left( \frac{r^{(\rho_{g_1} (g_1) - \varepsilon)}}{3} \right) < M_{g_1} (r)$$
i.e., $$\log \left( \frac{r^{(\rho_{g_1} (g_1) - \varepsilon)}}{3} \right) < \log M_{f_1 \pm f_2}^{-1} M_{g_1} (r)$$
i.e., $$(\rho_{g_1} (g_1) - \varepsilon) \log r + O(1) < \log M_{f_1 \pm f_2}^{-1} M_{g_1} (r)$$
i.e., $$(\rho_{g_1} (g_1) - \varepsilon) \log r + O(1) < \frac{\log M_{f_1 \pm f_2}^{-1} M_{g_1} (r)}{\log r}.$$Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\rho_{f_1 \pm f_2} (g_1) = \lim_{r \to \infty} \sup \frac{\log M_{f_1 \pm f_2}^{-1} M_{g_1} (r)}{\log r} \geq \rho_{f_i} (g_1).$$

Now without loss of genetality, we may consider that $\rho_{f_1} (g_1) < \rho_{f_2} (g_1)$ and $f = f_1 \pm f_2$. Then $\rho_{f} (g_1) \geq \rho_{f_1} (g_1)$. Further, $f_1 = (f \pm f_2)$ and in this case we obtain that $\rho_{f_1} (g_1) \geq \min \{\rho_{f} (g_1), \rho_{f_2} (g_1)\}$ . As we assume that $\rho_{f_1} (g_1) < \rho_{f_2} (g_1)$, therefore we have $\rho_{f_1} (g_1) \geq \rho_{f} (g_1)$ and hence $\rho_{f} (g_1) = \rho_{f_1} (g_1) = \min \{\rho_{f_1} (g_1), \rho_{f_2} (g_1)\}$. Therefore, $\rho_{f_1 \pm f_2} (g_1) = \rho_{f_1} (g_1) \mid i = 1, 2$ provided $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_1)$. Thus the first part of the theorem follows.

**Case II.** If $\lambda_{f_1} (g_1 \pm g_2) = 0$ then $\lambda_{f_1} (g_1 \pm g_2) \leq \lambda_{f_1} (g_i)$ is obvious. So we
suppose that $\lambda_{f_i} (g_1 \pm g_2) > 0$. We can clearly assume that $\lambda_{f_i} (g_i) \mid i = 1, 2$ is finite. Also suppose that $\lambda_{f_i} (g_k) \leq \lambda_{f_i} (g_i)$ where $k = i = 1, 2$ with $g_k \neq g_i$ and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$. Now in view of \([1], (4)\) and Lemma \([2]\) we get for a sequence of values of $r$ tending to infinity that

$$M_{g_1 \pm g_2} (r) < M_{g_1} (r) + M_{g_2} (r)$$

\[\text{i.e., } M_{g_1 \pm g_2} (r) < \sum_{k=1}^{2} M_{f_k} \left( r^{(\lambda_{f_k} (g_k) + \varepsilon)} \right)\]

\[\text{i.e., } M_{g_1 \pm g_2} (r) < 2M_{f_1} \left( r^{(\lambda_{f_1} (g_1) + \varepsilon)} \right)\]

\[\text{i.e., } M_{g_1 \pm g_2} (r) < M_{f_1} \left( 3r^{(\lambda_{f_1} (g_1) + \varepsilon)} \right)\]

\[\text{i.e., } \log M_{g_1 \pm g_2}^{-1} (r) < (\lambda_{f_1} (g_1) + \varepsilon) \log r + O(1)\]

\[\text{i.e., } \frac{\log M_{g_1 \pm g_2}^{-1} (r)}{\log r} < \frac{(\lambda_{f_1} (g_1) + \varepsilon) \log r + O(1)}{\log r}\]

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\lambda_{f_1} (g_1 \pm g_2) = \liminf_{r \to \infty} \frac{\log M_{g_1 \pm g_2}^{-1} (r)}{\log r} \leq \lambda_{f_1} (g_i) .$$

Further without loss of generality, let $\lambda_{f_1} (g_1) < \lambda_{f_1} (g_2)$ and $g = g_1 \pm g_2$. Then $\lambda_{f_1} (g) \leq \lambda_{f_1} (g_2)$ . Further, $g_2 = \pm (g - g_1)$ and in this case we obtain that $\lambda_{f_1} (g_2) \leq \max \{ \lambda_{f_1} (g), \lambda_{f_1} (g_1) \}$. As we assume that $\lambda_{f_1} (g_1) < \lambda_{f_1} (g_2)$, therefore we have $\lambda_{f_1} (g_2) \leq \lambda_{f_1} (g)$ and hence $\lambda_{f_1} (g) = \lambda_{f_1} (g_2) = \max \{ \lambda_{f_1} (g_1), \lambda_{f_1} (g_2) \}$. Therefore, $\lambda_{f_1} (g_1 \pm g_2) = \lambda_{f_1} (g_i) \mid i = 1, 2$ provided $\lambda_{f_1} (g_1) \neq \lambda_{f_1} (g_2)$. Thus the second part of the theorem is established.

In the line of Theorem A, Theorem C and Theorem \([3]\) one may state the following theorem without its proof:

Let $f_1, f_2, g_1$ and $g_2$ be any four entire functions. Then

\[(i) \quad \rho_{f_1 \pm f_2} (g_1 \pm g_2) \leq \min \{ \min \{ \rho_{f_1} (g_1), \rho_{f_2} (g_1) \}, \min \{ \rho_{f_1} (g_2), \rho_{f_2} (g_2) \} \}\]

when $\rho_{f_1} (g_1) \neq \rho_{f_2} (g_1)$, $\rho_{f_1} (g_2) \neq \rho_{f_2} (g_2)$ and $g_1$ and $g_2$ are both of regular relative growth with respect to at least any one of $f_1$ or $f_2$. The sign of equality holds when $\min \{ \rho_{f_1} (g_1), \rho_{f_2} (g_1) \} \neq \min \{ \rho_{f_1} (g_2), \rho_{f_2} (g_2) \}$ and

\[(ii) \quad \lambda_{f_1 \pm f_2} (g_1 \pm g_2) \geq \max \{ \max \{ \lambda_{f_1} (g_1), \lambda_{f_2} (g_1) \}, \max \{ \lambda_{f_1} (g_2), \lambda_{f_2} (g_2) \} \}\]

when $\lambda_{f_1} (g_1) \neq \lambda_{f_2} (g_1)$, $\lambda_{f_1} (g_2) \neq \lambda_{f_2} (g_2)$ and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$ and $f_2$ respectively. The sign of equality holds when $\max \{ \lambda_{f_1} (g_1), \lambda_{f_2} (g_1) \} \neq \max \{ \lambda_{f_1} (g_2), \lambda_{f_2} (g_2) \}$.
Let \( f_1, f_2, g_1 \) and \( g_2 \) be any four entire functions. Then

(i) \[
\rho_{f_1,f_2}(g_1) \geq \rho_{f_1}(g_1)
\]

where \( \rho_{f_1}(g_1) = \min \{\rho_{f_k}(g_1) \mid k = 1, 2\} \), \( g_1 \) has the Property (A) and also \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \). The sign of equality holds when \( \rho_{f_1}(g_1) \neq \rho_{f_2}(g_1) \). Similar results hold for the quotient \( \frac{g_1}{g_2} \) provided \( \frac{f_1}{f_2} \) is entire and

(ii) \[
\lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_i)
\]

where \( \lambda_{f_1}(g_i) = \max \{\lambda_{f_1}(g_k) \mid k = 1, 2\} \), \( f_1 \) has the Property (A) and at least \( g_1 \) or \( g_2 \) is of regular relative growth with respect to \( f_1 \). The sign of equality holds when \( \lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2) \). Similar results hold for the quotient \( \frac{g_1}{g_2} \) provided \( \frac{g_1}{g_2} \) is entire.

For any two entire functions \( h_1 \) and \( h_2 \), we have for all sufficiently large values of \( r \) that

\[
T_{h_1,h_2}(r) \leq T_{h_1}(r) + T_{h_2}(r).
\]

**Case I.** By Lemma 2 \( g_1 \) is transcendental. Suppose that \( \rho_{f_1,f_2}(g_1) < \infty \). Otherwise if \( \rho_{f_1,f_2}(g_1) = \infty \) then the result is obvious. We can clearly assume that \( \rho_{f_i}(g_1) \mid i = 1, 2 \) is finite. Also suppose that \( \rho_{f_i}(g_1) \leq \rho_{f_k}(g_1) \) where \( k = i = 1, 2 \) with \( f_i \neq f_k \) and \( g_1 \) is of regular relative growth with respect to at least any one of \( f_1 \) or \( f_2 \). Now for a sequence of values of \( r \) tending to infinity and for any \( \delta > 1 \), we get from (2), (3), (5) (considering \( h = f \) in (5)) and also in view of Lemma 2 and Lemma 2:

\[
\frac{1}{3} \log M_{f_1,f_2} \left( \frac{r}{2} \right) \leq \log M_{f_1}(r) + \log M_{f_2}(r)
\]

i.e.,
\[
\frac{1}{3} \log M_{f_1,f_2} \left( \frac{r}{2} \right) \leq \sum_{k=1}^{2} \log M_{g_1} \left( \frac{1}{r^{\rho_{f_k}(g_1)-\delta}} \right)
\]

i.e.,
\[
\frac{1}{3} \log M_{f_1,f_2} \left( \frac{r}{2} \right) \leq 2 \log M_{g_1} \left( \frac{1}{r^{\rho_{f_k}(g_1)-\delta}} \right)
\]

i.e.,
\[
\log M_{f_1,f_2} \left( \frac{r}{2} \right) \leq 6 \log M_{g_1} \left( \frac{1}{r^{\rho_{f_2}(g_1)-\delta}} \right)
\]

i.e.,
\[
M_{f_1,f_2} \left( \frac{r}{2} \right) \leq \left[ M_{g_1} \left( \frac{1}{r^{\rho_{f_2}(g_1)-\delta}} \right) \right]^6
\]

i.e.,
\[
M_{f_1,f_2} \left( \frac{r}{2} \right) \leq M_{g_1} \left( \frac{1}{r^{\rho_{f_2}(g_1)-\delta}} \right)
\]
for any $\delta > 0$ with respect to $f = \lambda$.

Case II. By Lemma 2, $g_1$ is transcendental. If \( \lambda_{f_1} (g_1 \cdot g_2) = 0 \) then \( \lambda_{f_1} (g_1 \cdot g_2) \leq \lambda_{f_1} (g_i) \) is obvious. So we suppose that \( \lambda_{f_1} (g_1 \cdot g_2) > 0 \). We can clearly assume that \( \lambda_{f_1} (g_i) \mid i = 1, 2 \) is finite. Also suppose that \( \lambda_{f_2} (g_k) \leq \lambda_{f_1} (g_i) \) where $k = i = 1, 2$ with $g_k \neq g_i$ and at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$. Now for a sequence of values of $r$ tending to infinity and for any $\delta > 1$, we obtain from (1), (4), (5) (considering $h = g$ in (5)) and in
view of Lemma 2 and Lemma 2.

\[
\frac{1}{3} \log M_{g_1, g_2} \left( \frac{r}{2} \right) \leq \log M_{g_1} (r) + \log M_{g_2} (r)
\]

i.e., \[
\frac{1}{3} \log M_{g_1, g_2} \left( \frac{r}{2} \right) \leq \sum_{k=1}^{2} \log M_{f_k} \left( r^{(\lambda_{\rho f_1}(g_k)+\varepsilon)} \right)
\]

i.e., \[
\frac{1}{3} \log M_{g_1, g_2} \left( \frac{r}{2} \right) \leq 2 \log M_{f_1} \left( r^{(\lambda_{\rho f_1}(g_k)+\varepsilon)} \right)
\]

i.e., \[
\log M_{g_1, g_2} \left( \frac{r}{2} \right) \leq 6 \log M_{f_1} \left( r^{(\lambda_{\rho f_1}(g_k)+\varepsilon)} \right)
\]

i.e., \[
M_{g_1, g_2} \left( \frac{r}{2} \right) \leq M_{f_1} \left( r^{(\lambda_{\rho f_1}(g_k)+\varepsilon)} \right)^6
\]

i.e., \[
M_{g_1, g_2} \left( \frac{r}{2} \right) \leq M_{f_1} \left( r^{(\lambda_{\rho f_1}(g_k)+\varepsilon)} \right)
\]

i.e., \[
\log M_{f_1}^{-1} M_{g_1, g_2} \left( \frac{r}{2} \right) \leq \delta (\lambda_{f_1} (g_k) + \varepsilon) \log r
\]

i.e., \[
\frac{\log M_{f_1}^{-1} M_{g_1, g_2} \left( \frac{r}{2} \right)}{\log r} \leq \frac{\delta (\lambda_{f_1} (g_k) + \varepsilon) \log r}{\log r + O(1)}.
\]

As \(\varepsilon > 0\) is arbitrary, we get from above by letting \(\delta \to 1+\),

\[
\lambda_{f_1} (g_1 \cdot g_2) = \liminf_{r \to \infty} \frac{\log M_{f_1}^{-1} M_{g_1} (r)}{\log r} \leq \lambda_{f_1} (g_1).
\]

Moreover without loss of any genetality, let \(\lambda_{f_1} (g_1) < \lambda_{f_1} (g_2)\) and \(g = g_1 \cdot g_2\). Then \(\lambda_{f_1} (g) \leq \lambda_{f_1} (g_2)\). Further, \(g_2 = \frac{g}{g_1}\) and \(T_{g_2} (r) = T_{g_1} (r) + O(1)\). Therefore \(T_{g_2} (r) \leq T_g (r) + T_{g_1} (r) + O(1)\), and in this case we obtain that \(\lambda_{f_1} (g_2) \leq \max \{\lambda_{f_1} (g), \lambda_{f_1} (g_1)\}\). As we assume that \(\lambda_{f_1} (g_1) < \lambda_{f_1} (g_2)\), therefore we have \(\lambda_{f_1} (g_2) \leq \lambda_{f_1} (g)\) and hence \(\lambda_{f_1} (g) = \lambda_{f_1} (g_2) = \max \{\lambda_{f_1} (g_1), \lambda_{f_1} (g_2)\}\). Therefore, \(\lambda_{f_1} (g_1 \cdot g_2) = \lambda_{f_1} (g) \mid i = 1, 2\) provided \(\lambda_{f_1} (g_1) \neq \lambda_{f_1} (g_2)\). Now let \(g = \frac{g_1}{g_2}\) where \(g_1, g_2, g\) are all entires and suppose that \(\lambda_{f_1} (g_1) \leq \lambda_{f_1} (g_2)\). We have \(g_1 = g \cdot g_2\). Therefore \(\lambda_{f_1} (g_1) = \lambda_{f_1} (g)\) if \(\lambda_{f_1} (g) > \lambda_{f_1} (g_2)\). So it follows that \(\lambda_{f_1} (g_1) > \lambda_{f_1} (g_2)\), which contradicts the hypothesis “\(\lambda_{f_1} (g_1) \leq \lambda_{f_1} (g_2)\)”. Hence \(\lambda_{f_1} (g) = \lambda_{f_1} \left( \frac{g_1}{g_2} \right) \leq \lambda_{f_1} (g_2) = \max \{\lambda_{f_1} (g_1), \lambda_{f_1} (g_2)\}\). Also suppose that \(\lambda_{f_1} (g_1) > \lambda_{f_1} (g_2)\). Then \(\lambda_{f_1} (g_1) = \max \{\lambda_{f_1} (g), \lambda_{f_1} (g_2)\} = \lambda_{f_1} (g_2)\), if \(\lambda_{f_1} (g) < \lambda_{f_1} (g_2)\), which is also a contradiction. Thus \(\lambda_{f_1} (g) = \lambda_{f_1} \left( \frac{g_1}{g_2} \right) = \max \{\lambda_{f_1} (g_1), \lambda_{f_1} (g_2)\}\). Therefore the second part of the theorem follows.

The proof of Theorem 3 is omitted because it can be carried in view of Theorem B, Theorem D and Theorem 3.
Let $f_1, f_2, g_1$ and $g_2$ be any four entire functions. Then

(a) $\rho_{f_1, f_2}(g_1 \cdot g_2) \leq \max \{\min \{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}, \min \{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}\}$,

(b) $\rho_{f_1, f_2}(\frac{g_1}{g_2}) \leq \max \{\min \{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}, \min \{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}\}$

when (i) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$, (ii) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ (iii) $f_1 \cdot f_2$, $g_1$ and $g_2$ have the Property (A) and (iv) $g_1$ and $g_2$ are both of regular relative growth with respect to at least any one of $f_1$ or $f_2$. The sign of equality holds when $\min \{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\} \neq \min \{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}$; and

(c) $\lambda_{f_1, f_2}(g_1 \cdot g_2) \geq \min \{\max \{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\}, \max \{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}\}$,

(d) $\lambda_{f_1, f_2}(\frac{g_1}{g_2}) \geq \min \{\max \{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\}, \max \{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}\}$

when (i) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$, (ii) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$, (iii) $g_1 \cdot g_2$, $f_1$ and $f_2$ have the Property (A) and (iv) at least $g_1$ or $g_2$ is of regular relative growth with respect to $f_1$ and $f_2$ respectively. The sign of equality holds when $\max \{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \max \{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}$.

References


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