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Some Extensions of Sum and Product Theorems on Relative order and Relative Lower Order of Entire Functions

Sanjib Kumar Datta

Department of Mathematics
University of Kalyani
P.O.- Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India,
e-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas

Rajbari, Rabindrapalli
R. N. Tagore Road, P.O.- Krishnagar
Dist-Nadia, PIN- 741101, West Bengal, India,
e-mail: tanmaybiswas_math@rediffmail.com

Pulakesh Sen

Department of Mathematics
D. N. C. College, P.O.- Aurangabad
Dist-Murshidabad, PIN-742201, West Bengal, India
e-mail: psendnc2011@gmail.com

Abstract

Some basic properties of relative order and relative lower order of entire functions have been discussed in this paper.

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1 Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding to f is defined as follows:

$$M_f(r) = \max|z| = r\,|f(z)| \ .$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. On the other hand, the Nevanlinna's Characteristic function of f denoted by $T_f(r)$ is defined as

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta$$

where

$$\log^+ x = \max(\log x, 0) \ \text{ for all } x \geqslant 0 \ .$$

For any two given entire functions f and g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The *order* of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the *exponential* function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} \ .$$

L. Bernal [1,2] introduced the definition of *relative order* of an entire function g with respect to an entire function f denoted by $\rho_f(g)$ to avoid comparing growth just with $\exp z$ which is as follows:

$$\begin{aligned}\rho_f(g) &= \inf\{\mu > 0 : M_g(r) < M_f(r^\mu) \ \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log M_f^{-1} M_g(r)}{\log r}.\end{aligned}$$

Definitely the above definition coincides with the classical one [10] if $f(z) = \exp z$.

Similarly, one can define the relative lower order of g with respect to f, denoted by $\lambda_f(g)$ as follows :

$$\lambda_f(g) = \liminf_{r\to\infty} \frac{\log M_f^{-1} M_g(r)}{\log r}.$$

An entire function g is said to be of regular relative growth with respect to f if its relative order with respect to f coincides with its relative lower order with respect to f.

During the past decades, several authors (see [5],[6],[7]) made close investigations on the properties of relative order of entire functions. In this connection the following definition is relevant:

[2] A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For exapmles of functions with or without the Property (A), one may see [2].

In this paper we wish to investigate some basic properties of relative order and relative lower order of entire functions under somewhat different conditions. We do not explain the standard definitions and notations of the theory of entire functions as those are available in [11].

2 Lemmas

In this section we present some lemmas which will be needed in the sequel.

[2] Suppose f be an entire function and α, β are such that $\alpha > 1$ and $0 < \beta < \alpha$. Then

$$M_f(\alpha r) > \beta M_f(r).$$

[2] Let f be an entire function satisfying the Property (A). Then for any positive integer n and for all sufficiently large r,

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

[8] Every entire function f satisfying the Property (A) is transcendental.

[9] Let f be an entire function. Then for all sufficiently large values of r,

$$T_f(r) \leq \log M_f(r) \leq 3T_f(2r) \quad \{\text{cf. [9], p. 18}\} .$$

3 Main Results

In this section we present the main results of the paper. First we recall related four theorems which are needed in order to prove our results.

Theorem A. [2] Let f_1, g_1 and g_2 be any three entire functions. Then

$$\rho_{f_1}(g_1 \pm g_2) \leq \rho_{f_1}(g_i)$$

where $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k = i = 1, 2\}$. The sign of equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$.

Theorem B. {[2],[9]} Let f_1, g_1 and g_2 be any three entire functions. Then

$$\rho_{f_1}(g_1 \cdot g_2) \leq \rho_{f_1}(g_i)$$

where $\rho_{f_1}(g_i) = \max\{\rho_{f_1}(g_k) \mid k = i = 1,2\}$. The sign of equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_1}(g_2)$. Similar results hold for the quotient $\frac{g_1}{g_2}$ provided $\frac{g_1}{g_2}$ is entire.
Theorem C. [3] Let f_1, f_2 and g_1 be any three entire functions. Then

$$\lambda_{f_1 \pm f_2}(g_1) \geq \lambda_{f_i}(g_1)$$

where $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = i = 1,2\}$. The sign of equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$.
Theorem D. [3] Let f_1, f_2 and g_1 be any three entire functions. Then

$$\lambda_{f_1 \cdot f_2}(g_1) \geq \lambda_{f_i}(g_1)$$

where $\lambda_{f_i}(g_1) = \min\{\lambda_{f_k}(g_1) \mid k = i = 1,2\}$. The sign of equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$. Similar results hold for the quotient $\frac{f_1}{f_2}$ provided $\frac{f_1}{f_2}$ is entire.
Now we prove the following results of the paper:

Let f_1, f_2, g_1 and g_2 be any four entire functions. Then
(i)

$$\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_i}(g_1)$$

where $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k = i = 1,2\}$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2. The sign of equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$; and
(ii)

$$\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i)$$

where $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k = i = 1,2\}$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. The sign of equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$.

From the definition of relative order and relative lower order of entire functions, we have for all sufficiently large values of r that

$$M_{g_k}(r) \leq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)+\varepsilon\right)}\right), \tag{1}$$

$$\begin{aligned} M_{g_k}(r) &\geq M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)-\varepsilon\right)}\right) \\ i.e.,\ M_{f_k}(r) &\leq M_{g_k}\left(r^{\frac{1}{\lambda_{f_k}(g_k)-\varepsilon}}\right), \end{aligned} \tag{2}$$

and also for a sequence values of r tending to infinity we get that

$$\begin{aligned} M_{g_k}(r) &\geq M_{f_k}\left(r^{\left(\rho_{f_k}(g_k)-\varepsilon\right)}\right) \\ i.e.,\ M_{f_k}(r) &\leq M_{g_k}\left(r^{\frac{1}{\rho_{f_k}(g_k)-\varepsilon}}\right), \end{aligned} \tag{3}$$

$$M_{g_k}(r) \leq M_{f_k}\left(r^{\left(\lambda_{f_k}(g_k)+\varepsilon\right)}\right) \tag{4}$$

where $\varepsilon\,(>0)$ is any arbitrary positive number and $i = 1, 2$.

Case I. If $\rho_{f_1 \pm f_2}(g_1) = \infty$ then $\rho_{f_1 \pm f_2}(g_1) \geq \rho_{f_i}(g_1)$ is obvious. So we suppose that $\rho_{f_1 \pm f_2}(g_1) < \infty$. We can clearly assume that $\rho_{f_i}(g_1) \mid i = 1, 2$ is finite. Also suppose that $\rho_{f_i}(g_1) \leq \rho_{f_k}(g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2. Now in view of (2), (3) and Lemma 2, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} M_{f_1 \pm f_2}(r) &< M_{f_1}(r) + M_{f_2}(r) \\ i.e.,\ M_{f_1 \pm f_2}(r) &< \sum_{k=1}^{2} M_{g_1}\left(r^{\frac{1}{\left(\rho_{f_k}(g_1)-\varepsilon\right)}}\right) \\ i.e.,\ M_{f_1 \pm f_2}(r) &< 2M_{g_1}\left(r^{\frac{1}{\left(\rho_{f_i}(g_1)-\varepsilon\right)}}\right) \end{aligned}$$

$$\begin{aligned} i.e.,\ M_{f_1 \pm f_2}\left(r^{\left(\rho_{f_i}(g_1)-\varepsilon\right)}\right) &< 2M_{g_1}(r) \\ i.e.,\ M_{f_1 \pm f_2}\left(\frac{r^{\left(\rho_{f_i}(g_1)-\varepsilon\right)}}{3}\right) &< M_{g_1}(r) \end{aligned}$$

$$\begin{aligned} i.e.,\ \log\left(\frac{r^{\left(\rho_{f_i}(g_1)-\varepsilon\right)}}{3}\right) &< \log M_{f_1 \pm f_2}^{-1} M_{g_1}(r) \\ i.e.,\ \left(\rho_{f_i}(g_1)-\varepsilon\right)\log r + O(1) &< \log M_{f_1 \pm f_2}^{-1} M_{g_1}(r) \\ i.e.,\ \left(\rho_{f_i}(g_1)-\varepsilon\right) + \frac{O(1)}{\log r} &< \frac{\log M_{f_1 \pm f_2}^{-1} M_{g_1}(r)}{\log r}\ . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\rho_{f_1 \pm f_2}(g_1) = \limsup_{r \to \infty} \frac{\log M_{f_1 \pm f_2}^{-1} M_{g_1}(r)}{\log r} \geq \rho_{f_i}(g_1)\,.$$

Now without loss of genetality, we may consider that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$ and $f = f_1 \pm f_2$. Then $\rho_f(g_1) \geq \rho_{f_1}(g_1)$. Further, $f_1 = (f \pm f_2)$ and in this case we obtain that $\rho_{f_1}(g_1) \geq \min\left\{\rho_f(g_1), \rho_{f_2}(g_1)\right\}$. As we assum that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$, therefore we have $\rho_{f_1}(g_1) \geq \rho_f(g_1)$ and hence $\rho_f(g_1) = \rho_{f_1}(g_1) = \min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}$. Therefore, $\rho_{f_1 \pm f_2}(g_1) = \rho_{f_i}(g_1) \mid i = 1, 2$ provided $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$. Thus the first part of the theorem follows.

Case II. If $\lambda_{f_1}(g_1 \pm g_2) = 0$ then $\lambda_{f_1}(g_1 \pm g_2) \leq \lambda_{f_1}(g_i)$ is obvious. So we suppose that $\lambda_{f_1}(g_1 \pm g_2) > 0$. We can clearly assume that $\lambda_{f_1}(g_i) \mid i = 1, 2$ is finite. Also suppose that $\lambda_{f_1}(g_k) \leq \lambda_{f_1}(g_i)$ where $k = i = 1, 2$ with $g_k \neq g_i$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Now in view of (1), (4) and Lemma 2, we get for a sequence of values of r tending to infinity that

$$\begin{aligned}
M_{g_1 \pm g_2}(r) &< M_{g_1}(r) + M_{g_2}(r) \\
i.e.,\ M_{g_1 \pm g_2}(r) &< \sum_{k=1}^{2} M_{f_1}\left(r^{\left(\lambda\rho_{f_1}(g_k)+\varepsilon\right)}\right) \\
i.e.,\ M_{g_1 \pm g_2}(r) &< 2M_{f_1}\left(r^{\left(\lambda_{f_1}(g_i)+\varepsilon\right)}\right) \\
i.e.,\ M_{g_1 \pm g_2}(r) &< M_{f_1}\left(3r^{\left(\lambda_{f_1}(g_i)+\varepsilon\right)}\right)
\end{aligned}$$

$$\begin{aligned}
i.e.,\ \log M_{f_1}^{-1} M_{g_1 \pm g_2}(r) &< (\lambda_{f_1}(g_i) + \varepsilon)\log r + O(1) \\
i.e.,\ \frac{\log M_{f_1}^{-1} M_{g_1 \pm g_2}(r)}{\log r} &< \frac{(\lambda_{f_1}(g_i) + \varepsilon)\log r + O(1)}{\log r}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\lambda_{f_1}(g_1 \pm g_2) = \liminf_{r\to\infty} \frac{\log M_{f_1}^{-1} M_{g_1 \pm g_2}(r)}{\log r} \leq \lambda_{f_1}(g_i) \ .$$

Further without loss of genetality, let $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2)$ and $g = g_1 \pm g_2$. Then $\lambda_{f_1}(g) \leq \lambda_{f_1}(g_2)$. Further, $g_2 = \pm(g - g_1)$ and in this case we obtain that $\lambda_{f_1}(g_2) \leq \max\{\lambda_{f_1}(g), \lambda_{f_1}(g_1)\}$. As we assume that $\lambda_{f_1}(g_1) < \lambda_{f_1}(g_2)$, therefore we have $\lambda_{f_1}(g_2) \leq \lambda_{f_1}(g)$ and hence $\lambda_{f_1}(g) = \lambda_{f_1}(g_2) = \max\{\lambda_{f_1}(g_1), \lambda_{f_1}(g_2)\}$. Therefore, $\lambda_{f_1}(g_1 \pm g_2) = \lambda_{f_1}(g_i) \mid i = 1, 2$ provided $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$. Thus the second part of the theorem is established.

In the line of Theorem A, Theorem C and Theorem 3, one may state the following theorem without its proof :

Let f_1, f_2, g_1 and g_2 be any four entire functions. Then

(i) $\rho_{f_1 \pm f_2}(g_1 \pm g_2) \leq \max\left[\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}, \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}\right]$

when $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$, $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ and g_1 and g_1 are both of regular relative growth with respect to at least any one of f_1 or f_2. The sign of equality holds when $\min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\} \neq \min\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\}$ and

(ii) $\lambda_{f_1 \pm f_2}(g_1 \pm g_2) \geq \min\left[\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\}, \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}\right]$

when $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$, $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$ and at least g_1 or g_2 is of regular relative growth with respect to f_1 and f_2 respectively. The sign of equality holds when $\max\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\} \neq \max\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\}$.

Let f_1, f_2, g_1 and g_2 be any four entire functions. Then
(i)

$$\rho_{f_1 \cdot f_2}(g_1) \geq \rho_{f_i}(g_1)$$

where $\rho_{f_i}(g_1) = \min\{\rho_{f_k}(g_1) \mid k = i = 1, 2\}$, g_1 has the Property (A) and also g_1 is of regular relative growth with respect to at least any one of f_1 or f_2. The sign of equality holds when $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$. Similar results hold for the quotient $\frac{f_1}{f_2}$ provided $\frac{f_1}{f_2}$ is entire and
(ii)

$$\lambda_{f_1}(g_1 \cdot g_2) \leq \lambda_{f_1}(g_i)$$

where $\lambda_{f_1}(g_i) = \max\{\lambda_{f_1}(g_k) \mid k = i = 1, 2\}$, f_1 has the Property (A) and at least g_1 or g_2 is of regular relative growth with respect to f_1. The sign of equality holds when $\lambda_{f_1}(g_1) \neq \lambda_{f_1}(g_2)$. Similar results hold for the quotient $\frac{g_1}{g_2}$ provided $\frac{g_1}{g_2}$ is entire.

For any two entire functions h_1 and h_2, we have for all sufficiently large values of r that

$$T_{h_1 \cdot h_2}(r) \leq T_{h_1}(r) + T_{h_2}(r) \ . \tag{5}$$

Case I. By Lemma 2, g_1 is transcendental. Suppose that $\rho_{f_1 \cdot f_2}(g_1) < \infty$. Otherwise if $\rho_{f_1 \cdot f_2}(g_1) = \infty$ then the result is obvious. We can clearly assume that $\rho_{f_i}(g_1) \mid i = 1, 2$ is finite. Also suppose that $\rho_{f_i}(g_1) \leq \rho_{f_k}(g_1)$ where $k = i = 1, 2$ with $f_i \neq f_k$ and g_1 is of regular relative growth with respect to at least any one of f_1 or f_2,. Now for a sequence of values of r tending to infinity and for any $\delta > 1$, we get from (2), (3), (5) (considering $h = f$ in (5)) and also in view of Lemma 2 and Lemma 2,

$$\begin{aligned}
\frac{1}{3} \log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq \log M_{f_1}(r) + \log M_{f_2}(r) \\
\textit{i.e.,}\ \frac{1}{3} \log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq \sum_{k=1}^{2} \log M_{g_1}\left(r^{\frac{1}{\rho_{f_k}(g_1) - \varepsilon}}\right) \\
\textit{i.e.,}\ \frac{1}{3} \log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq 2 \log M_{g_1}\left(r^{\frac{1}{\rho_{f_k}(g_1) - \varepsilon}}\right) \\
\textit{i.e.,}\ \log M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq 6 \log M_{g_1}\left(r^{\frac{1}{\rho_{f_i}(g_1) - \varepsilon}}\right)
\end{aligned}$$

$$\begin{aligned}
\textit{i.e.,}\ M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq \left[M_{g_1}\left(r^{\frac{1}{\rho_{f_i}(g_1) - \varepsilon}}\right)\right]^6 \\
\textit{i.e.,}\ M_{f_1 \cdot f_2}\left(\frac{r}{2}\right) &\leq M_{g_1}\left(r^{\frac{\delta}{\rho_{f_i}(g_1) - \varepsilon}}\right)
\end{aligned}$$

$$\begin{aligned} i.e.,\ M_{f_1\cdot f_2}\left(\frac{r^{\frac{\rho_{f_i}(g_1)-\varepsilon}{\delta}}}{2}\right) &\leq M_{g_1}(r) \\ i.e.,\ \log\left(\frac{r^{\frac{\rho_{f_i}(g_1)-\varepsilon}{\delta}}}{2}\right) &\leq \log M^{-1}_{f_1\cdot f_2}M_{g_1}(r) \end{aligned}$$

$$\begin{aligned} i.e.,\ \left(\frac{\rho_{f_i}(g_1)-\varepsilon}{\delta}\right)\log r + O(1) &\leq \log M^{-1}_{f_1\cdot f_2}M_{g_1}(r) \\ i.e.,\ \frac{\rho_{f_i}(g_1)}{\delta}-\frac{\varepsilon}{\delta}+\frac{O(1)}{\log r} &\leq \frac{\log M^{-1}_{f_1\cdot f_2}M_{g_1}(r)}{\log r}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain by letting $\delta \to 1+$,

$$\rho_{f_1\cdot f_2}(g_1) = \limsup_{r\to\infty}\frac{\log M^{-1}_{f_1\cdot f_2}M_{g_1}(r)}{\log r} \geq \rho_{f_i}(g_1)\ .$$

Now without loss of any genetality, we may consider that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$ and $f = f_1\cdot f_2$. Then $\rho_f(g_1) \geq \rho_{f_1}(g_1)$. Further, $f_1 = \frac{f}{f_2}$ and and $T_{f_2}(r) = T_{\frac{1}{f_2}}(r) + O(1)$. Therefore $T_{f_1}(r) \leq T_f(r) + T_{f_2}(r) + O(1)$, and in this case we obtain that $\rho_{f_1}(g_1) \geq \min\{\rho_f(g_1), \rho_{f_2}(g_1)\}$. As we assume that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$, so we have $\rho_{f_1}(g_1) \geq \rho_f(g_1)$ and hence $\rho_f(g_1) = \rho_{f_1}(g_1) = \min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}$. Therefore, $\rho_{f_1\cdot f_2}(g_1) = \rho_{f_i}(g_1) \mid i = 1,2$ provided $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$.
Further suppose that $f = \frac{f_1}{f_2}$ where $f_1,\ f_2,\ f$ are entires and let $\rho_{f_1}(g_1) \geq \rho_{f_2}(g_1)$. We have $f_1 = f\cdot f_2$. Therefore $\rho_{f_1}(g_1) = \rho_f(g_1)$ if $\rho_f(g_1) < \rho_{f_2}(g_1)$. So it follows that $\rho_{f_1}(g_1) < \rho_{f_2}(g_1)$, which contradicts the hypothesis "$\rho_{f_1}(g_1) \geq \rho_{f_2}(g_1)$". Hence $\rho_f(g_1) = \rho_{\frac{f_1}{f_2}}(g_1) \geq \rho_{f_2}(g_1)$ $= \min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}$. Also suppose that $\rho_{f_1}(g_1) > \rho_{f_2}(g_1)$. Then $\rho_{f_1}(g_1)$ $= \min\{\rho_f(g_1), \rho_{f_2}(g_1)\} = \rho_{f_2}(g_1)$, if $\rho_f(g_1) > \rho_{f_2}(g_1)$, which is also a contradiction. Thus $\rho_f(g_1) = \rho_{\frac{f_1}{f_2}}(g_1) = \min\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\}$. Thus the first part of the theorem is established.
Case II. By Lemma 2, g_1 is transcendental. If $\lambda_{f_1}(g_1\cdot g_2) = 0$ then $\lambda_{f_1}(g_1\cdot g_2) \leq \lambda_{f_1}(g_i)$ is obvious. So we suppose that $\lambda_{f_1}(g_1\cdot g_2) > 0$. We can clearly assume that $\lambda_{f_1}(g_i) \mid i = 1,2$ is finite. Also suppose that $\lambda_{f_1}(g_k) \leq \lambda_{f_1}(g_i)$ where $k = i = 1,2$ with $g_k \neq g_i$ and at least g_1 or g_2 is of regular relative growth with respect to f_1. Now for a sequence of values of r tending to infinity and for any $\delta > 1$, we obtain from (1), (4) , (5) (considering $h = g$ in (5)) and in

view of Lemma 2 and Lemma 2,

$$\begin{aligned}
\frac{1}{3}\log M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq \log M_{g_1}(r)+\log M_{g_2}(r)\\
\textit{i.e.,}\ \frac{1}{3}\log M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq \sum_{k=1}^{2}\log M_{f_1}\left(r^{\left(\lambda\rho_{f_1}(g_k)+\varepsilon\right)}\right)\\
\textit{i.e.,}\ \frac{1}{3}\log M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq 2\log M_{f_1}\left(r^{\left(\lambda\rho_{f_1}(g_k)+\varepsilon\right)}\right)\\
\textit{i.e.,}\ \log M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq 6\log M_{f_1}\left(r^{\left(\lambda\rho_{f_1}(g_k)+\varepsilon\right)}\right)
\end{aligned}$$

$$\begin{aligned}
\textit{i.e.,}\ M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq M_{f_1}\left[\left(r^{\left(\lambda_{f_1}(g_i)+\varepsilon\right)}\right)\right]^6\\
\textit{i.e.,}\ M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq M_{f_1}\left(r^{\delta\left(\lambda_{f_1}(g_i)+\varepsilon\right)}\right)
\end{aligned}$$

$$\begin{aligned}
\textit{i.e.,}\ \log M_{f_1}^{-1}M_{g_1\cdot g_2}\left(\frac{r}{2}\right) &\leq \delta\left(\lambda_{f_1}(g_i)+\varepsilon\right)\log r\\
\textit{i.e.,}\ \frac{\log M_{f_1}^{-1}M_{g_1\cdot g_2}\left(\frac{r}{2}\right)}{\log\left(\frac{r}{2}\right)} &\leq \frac{\delta\left(\lambda_{f_1}(g_i)+\varepsilon\right)\log r}{\log r+O(1)}.
\end{aligned}$$

As $\varepsilon>0$ is arbitrary, we get from above by letting $\delta\to 1+$,

$$\lambda_{f_1}(g_1\cdot g_2)=\liminf_{r\to\infty}\frac{\log M_{f_1\cdot f_2}^{-1}M_{g_1}(r)}{\log r}\leq\lambda_{f_1}(g_i)\ .$$

Moreover without loss of any genetality, let $\lambda_{f_1}(g_1)<\lambda_{f_1}(g_2)$ and $g=g_1\cdot g_2$. Then $\lambda_{f_1}(g)\leq\lambda_{f_1}(g_2)$. Further, $g_2=\frac{g}{g_1}$ and $T_{g_1}(r)=T_{\frac{1}{g_1}}(r)+O(1)$. Therefore $T_{g_2}(r)\leq T_g(r)+T_{g_1}(r)+O(1)$, and in this case we obtain that $\lambda_{f_1}(g_2)\leq\max\{\lambda_{f_1}(g),\lambda_{f_1}(g_1)\}$. As we assume that $\lambda_{f_1}(g_1)<\lambda_{f_1}(g_2)$, therefore we have $\lambda_{f_1}(g_2)\leq\lambda_{f_1}(g)$ and hence $\lambda_{f_1}(g)=\lambda_{f_1}(g_2)=\max\{\lambda_{f_1}(g_1),\lambda_{f_1}(g_2)\}$. Therefore, $\lambda_{f_1}(g_1\cdot g_2)=\lambda_{f_1}(g_i)\mid i=1,2$ provided $\lambda_{f_1}(g_1)\neq\lambda_{f_1}(g_2)$. Now let $g=\frac{g_1}{g_2}$ where g_1, g_2, g are all entires and suppose that $\lambda_{f_1}(g_1)\leq\lambda_{f_1}(g_2)$. We have $g_1=g\cdot g_2$. Therefore $\lambda_{f_1}(g_1)=\lambda_{f_1}(g)$ if $\lambda_{f_1}(g)>\lambda_{f_1}(g_2)$. So it follows that $\lambda_{f_1}(g_1)>\lambda_{f_1}(g_2)$, which contradicts the hypothesis "$\lambda_{f_1}(g_1)\leq\lambda_{f_1}(g_2)$". Hence $\lambda_{f_1}(g)=\lambda_{f_1}\left(\frac{g_1}{g_2}\right)\leq\lambda_{f_1}(g_2)=\max\{\lambda_{f_1}(g_1),\lambda\rho_{f_1}(g_2)\}$. Also suppose that $\lambda_{f_1}(g_1)>\lambda_{f_1}(g_2)$. Then $\lambda_{f_1}(g_1)=\max\{\lambda_{f_1}(g),\lambda_{f_1}(g_2)\}=\lambda_{f_1}(g_2)$, if $\lambda_{f_1}(g)<\lambda_{f_1}(g_2)$, which is also a contradiction. Thus $\lambda_{f_1}(g)=\lambda_{f_1}\left(\frac{g_1}{g_2}\right)=\max\{\lambda_{f_1}(g_1),\lambda\rho_{f_1}(g_2)\}$. Therefore the second part of the theorem follows.

The proof of Theorem 3 is omitted because it can be carried in view of Theorem B, Theorem D and Therorem 3.

Let f_1, f_2, g_1 and g_2 be any four entire functions. Then

(a) $\rho_{f_1 \cdot f_2}(g_1 \cdot g_2) \leq \max\left[\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}, \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}\right]$,

(b) $\rho_{\frac{f_1}{f_2}}\left(\dfrac{g_1}{g_2}\right) \leq \max\left[\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\}, \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}\right]$

when (i) $\rho_{f_1}(g_1) \neq \rho_{f_2}(g_1)$, (ii) $\rho_{f_1}(g_2) \neq \rho_{f_2}(g_2)$ (iii) $f_1 \cdot f_2$, g_1 and g_2 have the Property (A) and (iv) g_1 and g_1 are both of regular relative growth with respect to at least any one of f_1 or f_2. The sign of equality holds when $\min\left\{\rho_{f_1}(g_1), \rho_{f_2}(g_1)\right\} \neq \min\left\{\rho_{f_1}(g_2), \rho_{f_2}(g_2)\right\}$; and

(c) $\lambda_{f_1 \cdot f_2}(g_1 \cdot g_2) \geq \min\left[\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\}, \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}\right]$,

(d) $\lambda_{\frac{f_1}{f_2}}\left(\dfrac{g_1}{g_2}\right) \geq \min\left[\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\}, \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}\right]$

when (i) $\lambda_{f_1}(g_1) \neq \lambda_{f_2}(g_1)$, (ii) $\lambda_{f_1}(g_2) \neq \lambda_{f_2}(g_2)$, (iii) $g_1 \cdot g_2$, f_1 and f_2 have the Property (A) and (iv) at least g_1 or g_2 is of regular relative growth with respect to f_1 and f_2 respectively. The sign of equality holds when $\max\left\{\lambda_{f_1}(g_1), \lambda_{f_2}(g_1)\right\} \neq \max\left\{\lambda_{f_1}(g_2), \lambda_{f_2}(g_2)\right\}$.

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