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Multiple Solutions for Elliptic Problem with Singular Cylindrical Potential and Critical Exponent

Mohammed El Mokhtar Ould El Mokhtar*

Qassim University, College of Science, Departement of Mathematics, BO 6644, Bu- raidah: 51452, Kingdom of Saudi Arabia

Abstract

In the present paper, a quasilinear elliptic problem with singular cylindrical potential and critical exponent, is considered. By using the Nehari manifold and mountain pass theorem, the existence of at least four distinct solutions is obtained. The result depends crucially on the parameters a, b, m, s, λ and μ.

Keywords: Singular cylindrical potential; Concave term; Critical exponent; Nehari manifold; Mountain pass theorem

Introduction

In this paper, we consider the multiplicity results of nontrivial nonnegative solutions of the following problem $(\mathcal{P}_{\lambda,\mu})$

$$\begin{cases} -div\left(|y|^{-2a}\nabla u\right)-\mu|y|^{-2(a+1)}u = p|y|^{-bp}|u|^{p-2}u+\lambda|y|^{-c}|u|^{s-2}u, \text{in} \mathbb{R}^N, y\neq 0 \\ u\in\mathcal{D}_a^{1,2} \end{cases}$$

where each point x in $\mathbb{R}^N$ is written as a pair $(y,z)\in\mathbb{R}^m\times\mathbb{R}^{N-m}$ where m and N are integers such that $N\geq 3$ and m belongs to $\{1,...,N\}$, $-\infty<a<\sqrt{\bar{\mu}_m}$ with $\bar{\mu}_m=(m-2)^2/4$, $a\leq b<a+1, 1<s<2$, $p=2N/(N-2+2(b-a))$ is the critical Caffarelli-Kohn-Nirenberg exponent, $0<c=s(a+1)+N(1-s/2)$, $-\infty<\mu<\left(\sqrt{\bar{\mu}_m}-a\right)^2$, λ is a real parameter.

In recent years, many auteurs have paid much attention to the following singular elliptic problem, i.e., the case $a=b=c=0, s=2$ in $(\mathcal{P}_{\lambda,\mu})$,

$$\begin{cases} -\Delta u-\mu|x|^{-2}u = |u|^{p-2}u+\lambda u, \text{in}\Omega \\ u=0 \qquad \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 3)$, $0\in\Omega, \lambda>0, 0\leq\mu<\bar{\mu}_N:=(N-2)^2/4$ and $2^*=2N/(N-2)$ is the critical Sobolev exponent, see [1-3] and references therein. The quasilinear form of (1.2) is discussed in [4]. Some results are already available for $(\mathcal{P}_{\lambda,\mu})$ in the case $m=N$. Wang and Zhou [5] proved that there exist at least two solutions for $(\mathcal{P}_{\lambda,\mu})$ with $a=0$, $0<\mu\leq\bar{\mu}_N=(N-2)^2/4$, Bouchekif and Matallah [6] showed the existence of two solutions of $(\mathcal{P}_{\lambda,\mu})$ under certain conditions on a weighted function h, when $0<\mu\leq\bar{\mu}_N$, $\lambda\in(0,\Lambda_*)$, $-\infty<a<(N-2)/2$ and $a\leq b<a+1$, with Λ_* a positive constant.

Concerning existence results in the case $m<N$, we cite [7-9] and the references therein. Musina [9] considered $(\mathcal{P}_{\lambda,\mu})$ with $-a/2$ instead of a and $\lambda=0$, also $(\mathcal{P}_{\lambda,\mu})$ with $a=0$, $b=0, \lambda=0$ and $a\neq 2-m$. She established the existence of a ground state solution when $2<m\leq N$ and $0<\mu<\bar{\mu}_{a,m}=((m-2+a)/2)^2$ for $(\mathcal{P}_{\lambda,\mu})$ with $-a/2$ instead of a and $\lambda=0$. She also showed that $(\mathcal{P}_{\lambda,\mu})$ with $a=0$, $b=0, \lambda=0$ does not admit ground state solutions. Badiale et al. [10] studied $(\mathcal{P}_{\lambda,\mu})$ with $a=0$, $b=0, \lambda=0$. They proved the existence of at least a nonzero nonnegative weak solution μ, satisfying $u(y,z)=u(|y|,z)$ when $2\leq m<N$ and $\mu<0$ Bouchekif and El Mokhtar [11] proved that $(\mathcal{P}_{\lambda,\mu})$ admits two distinct solutions when $2<m\leq N$, $b=N-p(N-2)/2$ with $p\in(2,2^*]$, $\mu<\bar{\mu}_{0,m}$, and $\lambda\in(0,\Lambda_*)$ where Λ_* is a positive constant. Terracini [12] proved that there is no positive solutions of $(\mathcal{P}_{\lambda,\mu})$ with $b=0$, $\lambda=0$ when $a\neq 0$ and $\mu<0$. The regular problem corresponding to $a=b=\mu=0$ has been considered on a regular bounded domain Ω by Tarantello [13]. She proved that, with a nonhomogeneous term $f\in H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, not identically zero and satisfying a suitable condition, the problem considered admits two distinct solutions.

Before formulating our results, we give some definitions and notations.

We denote by $\mathcal{D}_a^{1,2}=\mathcal{D}_a^{1,2}(\mathbb{R}^m\setminus\{0\}\times\mathbb{R}^{N-m})$ and $\mathcal{H}_\mu=\mathcal{H}_\mu(\mathbb{R}^m\setminus\{0\}\times\mathbb{R}^{N-m})$, the closure of $C_0^\infty(\mathbb{R}^m\setminus\{0\}\times\mathbb{R}^{N-m})$ with respect to the norms

$$\|u\|_{a,0}=\left(\int_{\mathbb{R}^N}|y|^{-2a}|\nabla u|^2dx\right)^{1/2}$$

and

$$\|u\|_{a,\mu}=\left(\int_{\mathbb{R}^N}\left(|y|^{-2a}|\nabla u|^2-\mu|y|^{-2(a+1)}|u|^2\right)dx\right)^{1/2},$$

respectively, with $\mu<\left(\sqrt{\bar{\mu}_m}-a\right)^2=((m-2(a+1))/2)^2$ for $m\neq 2(a+1)$.

From the Hardy-Sobolev-Maz'ya inequality, it is easy to see that the norm $\|u\|_{a,\mu}$ is equivalent to $\|u\|_{a,0}$. More explicitly, we have

$$\left(1-\left(\sqrt{\bar{\mu}_m}-a\right)^{-2}\mu^+\right)^{1/2}\|u\|_{0,a}\leq\|u\|_{\mu,a}\leq\left(1-\left(\sqrt{\bar{\mu}_m}-a\right)^{-2}\mu^-\right)^{1/2}\|u\|_{0,a},$$

with $\mu^+=\max(\mu,0)$ and $\mu^-=\min(\mu,0)$ for all $u\in\mathcal{H}_\mu$.

We list here a few integral inequalities.

The starting point for studying $(\mathcal{P}_{\lambda,\mu})$, is the Hardy-Sobolev-Maz'ya inequality that is particular to the cylindrical case $m<N$ and that was proved by Maz'ya in [8]. It states that there exists positive constant $C_{a,p}$ such that

$$C_{a,p}\left(\int_{\mathbb{R}^N}|y|^{-bp}|v|^p dx\right)^{2/p}\leq\int_{\mathbb{R}^N}\left(|y|^{-2a}|\nabla v|^2-\mu|y|^{-2(a+1)}v^2\right)dx, \tag{1}$$

for any $v\in C_c^\infty((\mathbb{R}^m\setminus\{0\})\times\mathbb{R}^{N-m})$.

The second one that we need is the Hardy inequality with cylindrical weights [9]. It states that

$$\left(\sqrt{\bar{\mu}_m}-a\right)^2\int_{\mathbb{R}^N}|y|^{-2(a+1)}v^2dx\leq\int_{\mathbb{R}^N}|y|^{-2a}|\nabla v|^2dx, \text{for all } v\in\mathcal{H}_\mu, \tag{2}$$

It is easy to see that (1) hold for any $u\in\mathcal{H}_\mu$ in the sense

***Corresponding author:** Mohammed El Mokhtar Ould El Mokhtar, Qassim University, College of Science, Departement of Mathematics, BO 6644, Bu- raidah: 51452, Kingdom of Saudi Arabia, Tel: +966 16 380 0050; E-mail: med.mokhtar66@yahoo.fr

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$$\left(\int_{\mathbb{R}^N} |y|^{-c} |u|^r \, dx\right)^{1/r} \le C_{a,r} \left(\int_{\mathbb{R}^N} |y|^{-2a} |\nabla v|^2 \, dx\right)^{1/2}, \tag{3}$$

where $C_{a,r}$ positive constant, $1 \le r \le 2N/(N-2)$, $c \le r(a+1)+N(1-r/2)$, and in [14], if $r < 2N/(N-2)$ the embedding $\mathcal{H}_\mu \rightarrowtail L_r\left(\mathbb{R}^N, |y|^{-c}\right)$ is compact, where $L_r\left(\mathbb{R}^N, |y|^{-c}\right)$ is the weighted L_r space with norm

$$|u|_{r,c} = \left(\int_{\mathbb{R}^N} |y|^{-c} |u|^r \, dx\right)^{1/r}.$$

Since our approach is variational, we define the functional J on $\mathcal{H}_\mu$ by

$$J(u) := (1/2)\|u\|_{\mu,a}^2 - F(u) - G(u),$$

with

$$F(u) := \int_{\mathbb{R}^N} |y|^{-bp} |u|^p \, dx, G(u) := (1/s)\int_{\mathbb{R}^N} |y|^{-c} \lambda |u|^s \, dx.$$

A point $u \in \mathcal{H}_\mu$ is a weak solution of the equation $\left(\mathcal{P}_{\lambda,\mu}\right)$ if it satisfies

$$\left\langle J'(u), \varphi \right\rangle := R(u)\varphi - S(u)\varphi - T(u)\varphi = 0, \text{for all} \varphi \in \mathcal{H}_\mu$$

with

$$R(u)\varphi := \int_{\mathbb{R}^N} \left(|y|^{-2a} (\nabla u \nabla \varphi) - \mu |y|^{-2(a+1)} (u\varphi)\right)$$

$$S(u)\varphi := p\int_{\mathbb{R}^N} |y|^{-bp} |u|^{p-1} \varphi$$

$$T(u)\varphi := \int_{\mathbb{R}^N} |y|^{-c} \lambda |u|^{s-1} \varphi.$$

Here $\langle \cdot,\cdot \rangle$ denotes the product in the duality $\mathcal{H}_\mu'$, $\mathcal{H}_\mu \left(\mathcal{H}_\mu' \text{dual of} \mathcal{H}_\mu\right)$.

Let

$$S_\mu := \inf_{u \in \mathcal{H}_\mu \setminus \{0\}} \frac{\|u\|_{\mu,a}^2}{\left(\int_{\mathbb{R}^N} |y|^{-bp} |u|^p \, dx\right)^{2/p}}$$

From [15], S_μ is achieved.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem $\left(\mathcal{P}_{\lambda,\mu}\right)$ on the constraint defined by the Nehari manifold, which are solutions of our system.

Let Λ_0 be positive number such that

$$\Lambda_0 := \left(C_{a,s}\right)^{-s} \left(S_\mu\right)^{p/2(p-2)} \left(\frac{p-2}{p-s}\right)^{1/(2-s)} \left[\left(\frac{2-s}{p(p-s)}\right)\right]^{1/(p-2)}.$$

Now we can state our main results.

Theorem 1

Assume that $-\infty < a < (m-2)/2$, $0 < c = s(a+1) + N(1-s/2)$, $-\infty < \mu < \left(\sqrt{\bar{\mu}_m} - a\right)^2$ and λ verifying $0 < \lambda < \Lambda_0$, then the system $\left(\mathcal{P}_{\lambda,\mu}\right)$ has at least one positive solution.

Theorem 2

In addition to the assumptions of the Theorem 1, if λ satisfying $0 < \lambda < (1/2)\Lambda_0$, then $\left(\mathcal{P}_{\lambda,\mu}\right)$ has at least two positive solutions.

Theorem 3

In addition to the assumptions of the Theorem 2, assuming $N \ge \max(3, 6(a-b+1))$, there exists a positive real Λ_1 such that, if λ satisfy $0 < \lambda < \min((1/2)\Lambda_0, \Lambda_1)$, then $\left(\mathcal{P}_{\lambda,\mu}\right)$ has at least two positive solution and two opposite solutions.

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2 In the last Section, we prove the Theorem 3.

Preliminaries

Definition 1 *Let $c \in \mathbb{R}$, E a Banach space and $I \in C^1(E, \mathbb{R})$.*

i) $(u_n)_n$ is a Palais-Smale sequence at level C (in short $(PS)_c$) in E for I if

$$I(u_n) = c + o_n(1) \text{and} I'(u_n) = o_n(1),$$

where $o_n(1)$ tends to o as n goes at infinity.

ii) We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence in E for I has a convergent subsequence.

Lemma 1 *Let X Banach space, and $J \in C^1(X, \mathbb{R})$ verifying the Palais -Smale condition. Suppose that $J(0) = 0$ and that:*

i) there exist $R > 0$, $r > 0$ such that if $\|u\| = R$, then $J(u) \ge r$;

ii) there exist $(u_0) \in X$ such that $\|u_0\| > R$ and $J(u_0) \le 0$;

let $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \left(J(\gamma(t))\right)$ where

$$\Gamma = \left\{\gamma \in C([0,1]; X) \text{such that} \gamma(0) = 0 \text{et} \gamma(1) = u_0\right\},$$

then C is critical value of J such that $c \ge r$.

Nehari manifold

It is well known that J is of class C^1 in $\mathcal{H}_\mu$ and the solutions of $\left(\mathcal{P}_{\lambda,\mu}\right)$ are the critical points of J which is not bounded below on $\mathcal{H}_\mu$. Consider the following Nehari manifold

$$\mathcal{N} = \left\{u \in \mathcal{H}_\mu \setminus \{0\} : \left\langle J'(u), u \right\rangle = 0\right\},$$

Thus, $u \in \mathcal{N}$ if and only if

$$\|u\|_{\mu,a}^2 - F(u) - G(u) = 0. \tag{4}$$

Note that N contains every nontrivial solution of the problem $\left(\mathcal{P}_{\lambda,\mu}\right)$. Moreover, we have the following results.

Lemma 2 J is coercive and bounded from below on N.

Proof. If $u \in \mathcal{N}$, then by (4) and the Hölder inequality, we deduce that

$$J(u) = ((p-2)/2p)\|u\|_{\mu,a}^2 - ((p-s)/ps)G(u) \tag{5}$$

$$\ge ((p-2)/2p)\|u\|_{\mu,a}^2$$

$$-\left(\frac{(p-s)}{ps}\right)\lambda^{1/(2-s)} \left(C_{a,p}\right)^s \|u\|_{\mu,a}^s.$$

Thus, J is coercive and bounded from below on N.

Define

$$\phi(u) = \left\langle J'(u), u \right\rangle.$$

Then, for $u \in \mathcal{N}$

$$\left\langle \phi'(u), u \right\rangle = 2\|u\|_{\mu,a}^2 - p^2 F(u) - sG(u) \tag{6}$$

$$= (2-s)\|u\|_{\mu,a}^2 - p(p-s)F(u)$$

$$= (p-s)G(u) - (p-2)\|u\|_{\mu,a}^2.$$

Now, we split N in three parts:

$$\mathcal{N}^+ = \left\{u \in \mathcal{N} : \left\langle \phi'(u), u \right\rangle > 0\right\}$$

$$\mathcal{N}^0 = \left\{ u \in \mathcal{N} : \left\langle \phi'(u), u \right\rangle = 0 \right\}$$

$$\mathcal{N}^- = \left\{ u \in \mathcal{N} : \left\langle \phi'(u), u \right\rangle < 0 \right\}.$$

We have the following results.

Lemma 3

Suppose that u_0 is a local minimizer for J on N. Then, if $u_0 \notin \mathcal{N}^0$, u_0 is a critical point of J.

Proof. If u_0 is a local minimizer for *J* on *N*, then u_0 is a solution of the optimization problem

$$\min_{\{u/\phi(u)=0\}} J(u).$$

Hence, there exists a Lagrange multipliers $\theta \in \mathbb{R}$ such that

$$J'(u_0) = \theta \phi'(u_0) \text{in} \mathcal{H}'$$

Thus,

$$\left\langle J'(u_0), u_0 \right\rangle = \theta \left\langle \phi'(u_0), u_0 \right\rangle.$$

But $\left\langle \phi'(u_0), u_0 \right\rangle \neq 0$, since $u_0 \notin \mathcal{N}^0$. Hence $\theta = 0$. This completes the proof.

Lemma 4

There exists a positive number Λ_0 such that for all λ, verifying

$$0 < \lambda < \Lambda_0,$$

we have $\mathcal{N}^0 = \varnothing$.

Proof. Let us reason by contradiction.

Suppose $\mathcal{N}^0 \neq \varnothing$ such that $0 < \lambda < \Lambda_0$. Then, by (6) and for $u \in \mathcal{N}^0$, we have

$$\|u\|^2_{\mu,a} = p(p-s)/(2-s) F(u) \tag{7}$$

$$= ((p-s)/(p-2)) G(u)$$

Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|_{\mu,a} \geq (S_\mu)^{p/2(p-2)} \left[(2-s)/p(p-s)\right]^{-1/(p-2)} \tag{8}$$

and

$$\|u\|_{\mu,a} \leq \left[\left(\frac{p-s}{p-2} \right)^{-1/(2-s)} \left(\lambda^{1/(2-s)}\right) \left(C_{a,s}\right)^s \right]. \tag{9}$$

From (8) and (9), we obtain $\lambda \geq \Lambda_0$, which contradicts an hypothesis.

Thus $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$. Define

$$c := \inf_{u \in \mathcal{N}} J(u), c^+ := \inf_{u \in \mathcal{N}^+} J(u) \text{and} c^- := \inf_{u \in \mathcal{N}^-} J(u).$$

For the sequel, we need the following Lemma.

Lemma 5

i) For all λ such that $0 < \lambda < \Lambda_0$, one has $c \leq c^+ < 0$.

ii) For all λ such that $0 < \lambda < (1/2)\Lambda_0$, one has

$$c^- > C_0 = C_0(\lambda, S_\mu)$$

$$= \left(\frac{(p-2)}{(2p)} \right) \left[\frac{(2-s)}{p(p-s)} \right]^{2/(p-2)} (S_\mu)^{p/(p-2)} +$$

$$- \left(\frac{(p-s)}{ps} \right) \left(\lambda^{1/(2-s)} \right) \left(C_{a,s} \right)^s.$$

Proof. *i)* Let $u \in \mathcal{N}^+$. By (6), we have

$$\left[(2-s)/p(p-1)\right] \|u\|^2_{\mu,a} > F(u)$$

and so

$$J(u) = (-1/2) \|u\|^2_{\mu,a} + (p-1) F(u)$$

$$< -\left[\frac{p(p-s) - 2(p-1)(2-s)}{2p(p-s)} \right] \|u\|^2_{\mu,a}.$$

We conclude that $c \leq c^+ < 0$.

ii) Let $u \in \mathcal{N}^-$. By (6), we get

$$\left[(2-s)/p(p-s)\right] \|u\|^2_{\mu,a} < F(u).$$

Moreover, by Sobolev embedding theorem, we have

$$F(u) \leq (S_\mu)^{-p/2} \|u\|^p_{\mu,a}.$$

This implies

$$\|u\|_{\mu,a} > (S_\mu)^{p/2(p-2)} \left[\frac{(2-s)}{p(p-s)} \right]^{-1/(p-2)}, \text{for all} u \in \mathcal{N}^-. \tag{10}$$

By (5), we get

$$J(u) \geq ((p-2)/2p) \|u\|^2_{\mu,a} +$$

$$- \left(\frac{(p-s)}{ps} \right) \lambda^{1/(2-s)} \left(C_{a,p} \right)^s \|u\|^s_{\mu,a}.$$

Thus, for all λ such that $0 < \lambda < (1/2)\Lambda_0$, we have $J(u) \geq C_0$.

For each $u \in \mathcal{H}$ with $\int_{\mathbb{R}^N} |y|^{-bp} |u|^p \, dx \neq 0$, we write

$$t_M := t_{\max}(u) = \left[\frac{(2-s)\|u\|^2_{\mu,a}}{p(p-s) \int_{\mathbb{R}^N} |y|^{-bp} |u|^p \, dx} \right]^{(2-s)/p(p-s)} > 0.$$

Lemma 6

Let λ real parameters such that $0 < \lambda < \Lambda_0$. For each $u \in \mathcal{H}$ with $\int_{\mathbb{R}^N} |y|^{-bp} |u|^p \, dx \neq 0$, one has the following:

There exist unique t^+ and t^- such that $0 < t^+ < t_M < t^-$, $(t^+ u) \in \mathcal{N}^+$, $t^- u \in \mathcal{N}^-$,

$$J(t^+ u) = \inf_{0 \leq t \leq t_M} J(tu) \text{and} J(t^- u) = \sup_{t \geq 0} J(tu).$$

Proof. With minor modifications, we refer to [16].

Proposition 1 [16]

i) For all λ such that $0 < \lambda < \Lambda_0$, there exists a $(PS)_{c^+}$ sequence in $\mathcal{N}^+$.

ii) For all λ such that $0 < \lambda < (1/2)\Lambda_0$, there exists a $(PS)_{c^-}$ sequence in $\mathcal{N}^-$.

Proof of Theorems 1

Now, taking as a starting point the work of Tarantello [13], we establish the existence of a local minimum for *J* on $\mathcal{N}^+$.

Proposition 2 *For all λ such that $0 < \lambda < \Lambda_0$, the functional J has a minimizer $^+ \in$ $^+$ and it satisfies:*

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(i) $J\left(u_0^+\right)=c=c^+$,

(ii) $\left(u_0^+\right)$ is a nontrivial solution of $\left(\mathcal{P}_{\lambda,\mu}\right)$.

Proof. If $0<\lambda<\Lambda_0$, then by Proposition 1(i) there exists a $(u_n)_n$ $(PS)_{c^+}$ sequence in $\mathcal{N}^+$, thus it bounded by Lemma 2. Then, there exists $u_0^+\in\mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$u_n \rightharpoonup u_0^+ \text{weaklyin}\mathcal{H} \tag{11}$$

$$u_n \rightharpoonup u_0^+ \text{weaklyin} L^p\left(\mathbb{R}^N,|y|^{-bp}\right)$$

$$u_n \to u_0^+ \text{stronglyin} L^s\left(\mathbb{R}^N,|y|^{-c}\right)$$

$$u_n \to u_0^+ \text{a.ein}\mathbb{R}^N$$

Thus, by (11), u_0^+ is a weak nontrivial solution of $\left(\mathcal{P}_{\lambda,\mu}\right)$. Now, we show that u_n converges to u_0^+ strongly in $\mathcal{H}$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\|_{\mu,a}<\liminf_{n\to\infty}\|u_n\|_{\mu,a}$ and we obtain

$$c\le J\left(u_0^+\right)=\left((p-2)/2p\right)\left\|u_0^+\right\|_{\mu,a}^2-\left((p-s)/ps\right)G\left(u_0^+\right)$$

$$<\liminf_{n\to\infty}J\left(u_n\right)=c.$$

We get a contradiction. Therefore, u_n converge to u_0^+ strongly in $\mathcal{H}$. Moreover, we have $u_0^+\in\mathcal{N}^+$. If not, then by Lemma 6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+u_0^+\right)\in\mathcal{N}^+$ and $\left(t^-u_0^+\right)\in\mathcal{N}^-$. In particular, we have $t_0^+ < t_0^-=1$. Since

$$\frac{d}{dt}J\left(tu_0^+\right)_{|t=t_0^+}=0\text{and}\frac{d^2}{dt^2}J\left(tu_0^+\right)_{|t=t_0^+}>0,$$

there exists $t_0^+<t^-\le t_0^-$ such that $J\left(t_0^+u_0^+\right)<J\left(t^-u_0^+\right)$. By Lemma 6, we get

$$J\left(t_0^+u_0^+\right)<J\left(t^-u_0^+\right)<J\left(t_0^-u_0^+\right)=J\left(u_0^+\right),$$

which contradicts the fact that $J\left(u_0^+\right)=c^+$. Since $J\left(u_0^+\right)=J\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right|\in\mathcal{N}^+$, then by Lemma [3], we may assume that u_0^+ is a nontrivial nonnegative solution of $\left(\mathcal{P}_{\lambda,\mu}\right)$. By the Harnack inequality, we conclude that $u_0^+>0$ and $v_0^+>0$, see for exanmple (7).

Proof of Theorem 2

Next, we establish the existence of a local minimum for J on $\mathcal{N}^-$. For this, we require the following Lemma.

Lemma 7

For all λ such that $0<\lambda<(1/2)\Lambda_0$, the functional J has a minimizer u_0^- in $\mathcal{N}^-$ and it satisfies:

(i) $J\left(u^-\right)=c^->0$,

(ii) u_0^- is a nontrivial solution of $\left(\mathcal{P}_{\lambda,\mu}\right)$ in $\mathcal{H}$.

Proof. If $0<\lambda<(1/2)\Lambda_0$, then by Proposition 1 i) there exists a $(u_n)_n$, $(PS)_{c^-}$ sequence in $\mathcal{N}^-$, thus it bounded by Lemma 2. Then, there exists $u_0^-\in\mathcal{H}$ and we can extract a subsequence which will denoted by $(u_n)_n$ such that

$$u_n \rightharpoonup u_0^- \text{weaklyin}\mathcal{H}$$

$$u_n \rightharpoonup u_0^- \text{weaklyin} L^p\left(\mathbb{R}^N,|y|^{-bp}\right)$$

$$u_n \to u_0^- \text{stronglyin} L^s\left(\mathbb{R}^N,|y|^{-c}\right)$$

$$u_n \to u_0^- \text{a.ein}\mathbb{R}^N$$

This implies

$$F\left(u_n\right)\to F\left(u_0^-\right),\text{as}n\text{goesto}\infty.$$

Moreover, by (6) we obtain

$$F\left(u_n\right)>A(p,s)\left\|u_n\right\|_{\mu,a}^2, \tag{12}$$

where, $A(p,s):=(2-s)/p(p-s)$. By (8) and (12) there exists a positive number

$$C_1:=\left[A(p,s)\right]^{p/(p-2)}\left(S_\mu\right)^{p/(p-2)},$$

such that

$$F\left(u_n\right)>C_1. \tag{13}$$

This implies that

$$F\left(u_0^-\right)\ge C_1.$$

Now, we prove that $(u_n)_n$ converges to u_0^- strongly in $\mathcal{H}$. Suppose otherwise. Then, either $\left\|u_0^-\right\|_{\mu,a}<\liminf_{n\to\infty}\|u_n\|_{\mu,a}$. By Lemma (6) there is a unique t_0^- such that $\left(t_0^-u_0^-\right)\in\mathcal{N}^-$. Since

$$u_n\in\mathcal{N}^-,J\left(u_n\right)\ge J\left(tu_n\right),\text{for all } t\ge 0,$$

we have

$$J\left(t_0^-u_0^-\right)<\lim_{n\to\infty}J\left(t_0^-u_n\right)\le\lim_{n\to\infty}J\left(u_n\right)=c^-,$$

and this is a contradiction. Hence,

$$\left(u_n\right)_n\to u_0^-\text{stronglyin}\mathcal{H}.$$

Thus,

$$J\left(u_n\right)\text{convergesto}J\left(u_0^-\right)=c^-\text{as}n\text{tendsto}+\infty.$$

Since $J\left(u_0^-\right)=J\left(\left|u_0^-\right|\right)$ and $u_0^-\in\mathcal{N}^-$, then by (13) and Lemma 3, we may assume that u_0^- is a nontrivial nonnegative solution of $\left(\mathcal{P}_{\lambda,\mu}\right)$. By the maximum principle, we conclude that $u_0^->0$.

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that $\left(\mathcal{P}_{\lambda,\mu}\right)$ has two positive solutions $u_0^+\in\mathcal{N}^+$ and $u_0^-\in\mathcal{N}^-$. Since $\mathcal{N}^+\cap\mathcal{N}^-=\varnothing$, this implies that u_0^+ and u_0^- are distinct [17,18].

Proof of Theorem 3

In this section, we consider the following Nehari submanifold of $\mathcal{N}$

$$\mathcal{N}_\rho=\left\{u\in\mathcal{H}\setminus\{0\}:\left\langle J'(u),u\right\rangle=0\text{and}\|u\|_{\mu,a}\ge\rho>0\right\}.$$

Thus, $u\in\mathcal{N}_\rho$ if and only if

$$\|u\|_{\mu,a}^2-pF(u)-G(u)=0\text{and}\|u\|_{\mu,a}\ge\rho>0.$$

Firsly, we need the following Lemmas

Lemma 8

Under the hypothesis of theorem 3, there exist ρ_0, $\Lambda_2>0$ such that $\mathcal{N}_\rho$ is nonempty for any $\lambda\in(0,\Lambda_2)$ and $\rho\in(0,\rho_0)$.

Proof. Fix $u_0\in\mathcal{H}\setminus\{0\}$ and let

$$g(t)=\left\langle J'\left(tu_0\right),tu_0\right\rangle$$

$$=t^2\left\|u_0\right\|_{\mu,a}^2-pt^pF\left(u_0\right)-tG\left(u_0\right).$$

Clearly $g(0)=0$ and $g(t)\to-\infty$ as $n\to+\infty$. Moreover, we have

$$g(1)=\|u_0\|_{\mu,a}^2-pF(u_0)-G(u_0)$$
$$\geq\left[\|u_0\|_{\mu,a}^2-p(S_\mu)^{-p/2}\|u_0\|_{\mu,a}^p\right]+$$
$$-\left(\lambda^{1/(2-s)}\right)\|u_0\|_{\mu,a}.$$

If $\|u_0\|_{\mu,a}\geq\rho>0$ for $0<\rho<\rho_0=(p(p-1))^{-1/(p-2)}(S_\mu)^{p/2(p-2)}$, then there exists

$$\Lambda_2:=\left[(p(p-1))(S_\mu)^{-p/2}\right]^{-1/(p-2)}-\Theta\times\Phi,$$

where

$$\Theta:=(p(p-1))^{p-1}(S_\mu)^{-(p)^2/2}$$

and

$$\Phi:=\left[(p(p-1))(S_\mu)^{-p/2}\right]^{-1/(p-2)}$$

and there exists $t_0>0$ such that $g(t_0)=0$. Thus, $(t_0u_0)\in\mathcal{N}_\rho$ and $\mathcal{N}_\rho$ is nonempty for any $\lambda\in(0,\Lambda_2)$.

Lemma 9

There exist V, Λ_1 positive reals such that

$$\left\langle\phi'(u),u\right\rangle<-V<0,\text{for}\,u\in\mathcal{N}_\rho,$$

and any λ verifying

$$0<\lambda<\min((1/2)\Lambda_0,\Lambda_1).$$

Proof. Let $u\in\mathcal{N}_\rho$, then by (4), (6) and the Holder inequality, allows us to write

$$\left\langle\phi'(u),u\right\rangle$$
$$\leq\|u_n\|_{\mu,a}^2\left[\left(\lambda^{1/(2-s)}\right)B(\rho,s)-(p-2)\right],$$

where $B(\rho,s):=(p-1)(C_{a,p})^s\rho^{s-2}$. Thus, if

$$0<\lambda<\Lambda_3=[(p-2)/B(\rho,s)],$$

and choosing $\Lambda_1:=\min(\Lambda_2,\Lambda_3)$ with Λ_2 defined in Lemma 8, then we obtain that

$$\left\langle\phi'(u),u\right\rangle<0,\text{for any}\,u\in\mathcal{N}_\rho. \tag{14}$$

Lemma 10

Suppose $N\geq\max(3,6(a-b+1))$ and $\int_{\mathbb{R}^N}|y|^{-bp}|u|^p\,dx>0$. Then, there exist r and η positive constants such that

i) we have

$$J(u)\geq\eta>0\text{ for }\|u\|_{\mu,a}=r.$$

ii) there exists $\sigma\in\mathcal{N}_\rho$ when $\|\sigma\|_{\mu,a}>r$, with $r=\|u\|_{\mu,a}$, such that $J(\sigma)\leq0$.

Proof. We can suppose that the minima of J are realized by (u_0^+) and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

i) By (6), (14) and the fact that $F(u)\leq(S_\mu)^{-p/2}\|u\|_{\mu,a}^p$, we get

$$J(u)\geq[(1/2)-(p-2)/(p-s)s]\|u\|_{\mu,a}^2-(S_\mu)^{-p/2}\|u\|_{\mu,a}^p,$$

Exploiting the function $l(x)=x(p-x)$ and if $N\geq\max(3,6(a-b+1))$, we obtain that $[(1/2)-(p-2)/(p-s)s]>0$ for $1<s<2$. Thus, there exist η, $r>0$ such that

$$J(u)\geq\eta>0\text{when}\,r=\|u\|_{\mu,a}\text{ small.}$$

ii) Let $t>0$, then we have for all $\phi\in\mathcal{N}_\rho$

$$J(t\phi):=(t^2/2)\|\phi\|_\mu^2-(t^p)F(\phi)-(t^s/s)G(\phi).$$

Letting $\sigma=t\phi$ for t large enough. Since

$$F(\phi):=\int_\Omega|y|^{-bp}|\phi|^p\,dx>0,$$

we obtain $J(\sigma)\leq0$. For t large enough we can ensure $\|\sigma\|_{\mu,a}>r$.

Let Γ and c defined by

$$\Gamma:=\left\{\gamma:[0,1]\to\mathcal{N}_\rho:\gamma(0)=u_0^-\text{and}\,\gamma(1)=u_0^+\right\}$$

and

$$c:=\inf_{\gamma\in\Pi}\max_{t\in[0,1]}\left(J(\gamma(t))\right).$$

Proof of Theorem 4

If

$$\lambda<\min((1/2)\Lambda_0,\Lambda_1),$$

then, by the Lemmas 2 and Proposition 1 (ii), J verifying the Palais -Smale condition in $\mathcal{N}_\rho$. Moreover, from the Lemmas 3, 9 and 10, there exists u_c such that

$$J(u_c)=c\,\text{and}\,u_c\in\mathcal{N}_\rho.$$

Thus u_c is the third solution of our system such that $u_c\neq u_0^+$ and $u_c\neq u_0^-$. Since $(\mathcal{P}_{\lambda,\mu})$ is odd with respect u, we obtain that $-u_c$ is also a solution of $(\mathcal{P}_{\lambda,\mu})$.

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