weak $\phi$-contraction on partial metric spaces
and existence of fixed points in partially ordered sets

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Abstract

In this manuscript, the notion of weak $\phi$-contraction is considered on partial metric space. It is shown that a self mapping $T$ on a complete partial metric space $X$ has a fixed point if they satisfied weak $\phi$-contraction.

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1 Introduction and Preliminaries

In 1992, Matthews [1, 2] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which $d(x, x)$ are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (See e.g. [3, 4, 5, 6, 16])

A partial metric space (See e.g.[1, 2]) is a pair $(X, p : S \times S \to \mathbb{R}^+)$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) such that

(PM1) $p(x, y) = p(y, x)$ (symmetry)
(PM2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)
(PM3) $p(x, x) \leq p(x, y)$ (small self-distances)
(PM4) $p(x, z) + p(y, y) \leq p(x, y) + p(y, z)$ (triangularity)
for all \( x, y, z \in X \). We use the abbreviation PMS for the partial metric space \((X, p)\).

Notice that for a partial metric \( p \) on \( X \), the function \( d_p : X \times X \to \mathbb{R}^+ \) given by
\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\]
is a (usual) metric on \( X \). Observe that each partial metric \( p \) on \( X \) generates a \( T_0 \) topology \( \tau_p \) on \( X \) with a base of the family of \( p \)-balls \( \{ B_p(x, \varepsilon) : x \in X, \varepsilon > 0 \} \), where \( B_p(x, \varepsilon) = \{ y \in X : p(x, y) < p(x, x) + \varepsilon \} \) for all \( x \in X \) and \( \varepsilon > 0 \). Similarly, closed \( p \)-ball is defined as \( B_p[x, \varepsilon] = \{ y \in X : p(x, y) \leq p(x, x) + \varepsilon \} \).

**Definition 1.1** (See e.g.
\[1, 2, 6\])

(i) A sequence \( \{x_n\} \) in a PMS \((X, p)\) converges to \( x \in X \) if and only if \( p(x, x) = \lim_{n \to \infty} p(x, x_n) \).

(ii) a sequence \( \{x_n\} \) in a PMS \((X, p)\) is called a Cauchy if and only if \( \lim_{n, m \to \infty} p(x_n, x_m) \) exists (and finite),

(iii) A PMS \((X, p)\) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \( X \) converges, with respect to \( \tau_p \), to a point \( x \in X \) such that \( p(x, x) = \lim_{n, m \to \infty} p(x_n, x_m) \).

(iv) A mapping \( f : X \to X \) is said to be continuous at \( x_0 \in X \), if for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon) \).

**Lemma 1.2** (See e.g.
\[1, 2, 6\])

(A) A sequence \( \{x_n\} \) is Cauchy in a PMS \((X, p)\) if and only if \( \{x_n\} \) is Cauchy in a metric space \((X, d_p)\).

(B) A PMS \((X, p)\) is complete if and only if a metric space \((X, d_p)\) is complete. Moreover,
\[
\lim_{n \to \infty} d_p(x, x_n) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n, m \to \infty} p(x_n, x_m) \tag{2}
\]

Boyd and Wong [7] introduced the notion of \( \Phi \)-contraction: a self mapping \( T \) on a metric space \( X \) is called \( \Phi \)-contraction if there exists an upper semi-continuous function \( \Phi : [0, \infty) \to [0, \infty) \) such that
\[
d(Tx, Ty) \leq \Phi(d(x, y)) \quad \text{for all } x, y \in X.
\]

Alber and Guerre-Delabriere [8], generalized the notion of \( \Phi \)-contraction by defining the notion of weak \( \phi \)-contraction for Hilbert spaces: A self mapping
Definition 2.1 (cf. [14]) Let \((X, \preceq)\) be a partially ordered set and \((X, p)\) a complete partial metric space. An operator \(T : X \to X\) is called a weak \(\phi\)-contraction if there exists a continuous, non-decreasing function \(\phi : [0, \infty) \to [0, \infty)\) with \(\phi(t) > 0\) for \(t \in (0, \infty)\) and \(\phi(0) = 0\), such that

\[
p(Tx, Ty) \leq p(x, y) - \phi(p(x, y)),
\]

for all \(x, y \in X\).

They also proved the existence of fixed points in Hilbert spaces. If one replaces Hilbert spaces with an arbitrary Banach spaces [8] still we have fixed points (See e.g. [9]). We should note that for a lower semi-continuous mapping \(\phi\), the function \(\Phi(u) = u - \phi(u)\) coincides with Boyd and Wong types.

In fixed point theory, \(\Phi\)-contraction and weak \(\phi\)-contraction have been studied by many authors, (See e.g., [10, 11, 12, 13, 14], also [15]). In this manuscript, by using weak \(\phi\)-contraction on a complete partial metric space we obtain a unique fixed point.

## 2 Main Results

**Theorem 2.2** Let \((X, \preceq)\) be a partially ordered set and \((X, p)\) a complete partial metric space. Suppose that \(T : X \to X\) is nondecreasing, continuous and weak \(\phi\)-contraction. If there exists an \(x_0 \in X\) with \(x_0 \preceq Tx_0\), \(T\) has a unique fixed point.

**Proof.** Let \(x_0 \in X\) and set \(x_{n+1} = Tx_n\). Notice that, if \(x_n = x_{n+1}\) for any \(n \geq 0\), then obviously \(T\) has a fixed point. Thus, suppose \(x_n \neq x_{n+1}\) for any \(n \geq 0\). Since \(x_0 \preceq Tx_0\), then

\[
x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots
\]

Due to (3), we have

\[
p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq p(x_n, x_{n+1}) - \phi(p(x_n, x_{n+1})).
\]

Define \(t_n = p(x_n, x_{n+1})\). Then one can obtain

\[
t_{n+1} \leq t_n - \phi(t_n) \leq t_n
\]
By triangle inequality we observe that \( \{t_n\} \) is a non-negative non-increasing sequence. Hence, \( \{t_n\} \) converges to \( L \) where \( L \geq 0 \). So there are two cases: \( L > 0 \) or \( L = 0 \). Assume that \( L > 0 \). Regarding that \( \phi \) is non-decreasing, we get \( 0 < \phi(L) \leq \phi(t_n) \).

Due to (5), we have \( t_{n+1} \leq t_n - \phi(t_n) \leq t_n - \phi(L) \) and so

\[
t_{n+2} \leq t_{n+1} - \phi(t_{n+1}) \leq t_n - \phi(t_n) - \phi(t_{n+1}) \leq t_n - 2\phi(L).
\]

Inductively we obtain \( t_{n+k} \leq t_n - k\phi(L) \) which is a contradiction for large enough \( k \in \mathbb{N} \). Hence we have \( L = 0 \). Thus, we have \( \lim_{n \to \infty} p(x_{n+1}, x_n) = 0 \).

Now, we show that \( \{x_n\} \) is a Cauchy sequence in \( (X, p) \). For this purpose, define \( s_n = \sup\{p(x_i, x_j) : i, j \geq n\} \). It is clear that the sequence \( \{s_n\} \) is decreasing. If \( \lim_{n \to \infty} s_n = 0 \), then \( \{x_n\} \) is a Cauchy sequence. So consider the other case: Suppose \( \lim_{n \to \infty} s_n = s > 0 \). One can choose \( \varepsilon \) small enough (e.g. \( \varepsilon < \frac{s}{10} \)) and a natural number \( N \) such that

\[
p(x_n, x_{n+1}) < \varepsilon, \text{ and } s_n < s + \varepsilon, \text{ for all } n \geq N.
\]

Regarding the definition of \( s_{N+1} \), there exist \( m, n \geq N + 1 \) such that

\[
s - \varepsilon < s_n - \varepsilon < p(x_m, x_n).
\]

By triangle inequality we observe that

\[
p(x_n, x_m) \leq p(x_n, x_{n-1}) + p(x_{n-1}, x_m) - p(x_{n-1}, x_{n-1}) \tag{8}
\]

\[
p(x_n, x_m) \leq p(x_n, x_{m-1}) + p(x_{m-1}, x_m) - p(x_{m-1}, x_{m-1}) \tag{9}
\]

\[
p(x_n-1, x_m) \leq p(x_{n-1}, x_{m-1}) + p(x_{m-1}, x_m) - p(x_{m-1}, x_{m-1}) \tag{10}
\]

Due to (7) and (6) the expression (8) and (9) yield that

\[
s - 2\varepsilon < p(x_{n-1}, x_m), \quad \text{and} \quad s - 2\varepsilon < p(x_n, x_{m-1}). \tag{11}
\]

Combining (10) and (11), we get that

\[
s - 3\varepsilon < p(x_{n-1}, x_{m-1}). \tag{12}
\]

Thus,

\[
p(x_n, x_m) = p(Tx_{n-1}, Tx_{m-1}) \leq p(x_{n-1}, x_{m-1}) - \phi(p(x_{n-1}, x_{m-1})) \leq p(x_{n-1}, x_{m-1}) - \phi(s) \tag{13}
\]

Regarding (7) and (12), the expression (13) implies that \( s_{N+1} < s_N - \phi(s) \) for small enough \( \varepsilon \). It is impossible. Hence \( s = 0 \). Notice that

\[
d_p(x_n, x_m) = 2p(x_n, x_m) - p(x_{n-1}, x_{n-1}) - \phi(p(x_{m-1}, x_{m-1})) \tag{14}
\]
Since $s = 0$, then $d_p(x_n, x_m) \to 0$. Therefore, the sequence $\{x_n\}$ is Cauchy in $(X, d_p)$.

Since $(X, p)$ is complete, by Lemma 2 $(X, d_p)$ is complete. and the sequence $\{x_n\}$ is convergent in $X$, say $z \in X$. Again by Lemma 2,

$$p(z, z) = \lim_{n \to \infty} p(x_n, z) = \lim_{n, m \to \infty} p(x_n, x_m) \quad (15)$$

Since $s = 0$, then by (15) we have $p(z, z) = 0$. We assert that $Tz = z$. Due to (PM4), we have

$$p(Tz, z) \leq p(Tz, Tx_n) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1})$$

Letting $n \to \infty$ and regarding the continuity of $\phi$, then (16) yields that $p(Tz, z) \leq 0$. Hence $Tz = z$.

Now we show $z$ is unique fixed point of $T$. Assume the contrary, that is, there exists $w \in X$ such that $z \neq w$ and $w = Tw$.

$$p(z, w) = p(Tz, Tw) \leq p(z, w) - \phi(p(z, w))$$

which is a contradiction. Thus $z$ is a unique fixed point of $T$. \qed

**Theorem 2.3** Let $(X, \preceq)$ be a partially ordered set and $(X, p)$ a complete partial metric space. Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for $t \in (0, \infty)$ and $\phi(0) = 0$. Suppose also that $T : X \to X$ is nondecreasing and satisfying

$$p(Tx, Ty) \leq p(x, y) - \phi(p(x, y)) \quad (17)$$

for any $x, y \in X$ with $x \prec (that is, x \preceq y$ and $x \neq y)$. Moreover the following condition is hold:

If $\{x_n\} \subset X$ is a increasing sequence with $x_n \to z$, then $x_n \prec z$, $\forall n$. \quad (18)

If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, $T$ has a fixed point.

**Proof.** As in the proof of Theorem 2.2, take $x_0 \in X$ and set $x_{n+1} = Tx_n$. If $x_n = x_{n+1}$ for any $n \geq 0$, then obviously $T$ has a fixed point. Thus, suppose $x_n \neq x_{n+1}$ for any $n \geq 0$. Since $x_0 \preceq Tx_0$, then

$$x_0 \prec x_1 \prec \cdots \prec x_n \preceq x_{n+1} \prec \cdots \quad (19)$$

As in the proof of Theorem 2.2, we observe that the sequence $\{x_n\}$ is Cauchy and thus it converges to $z \in X$. Hence, we have (as in the proof of Theorem 2.2)

$$p(z, z) = \lim_{n \to \infty} p(x_n, z) = \lim_{n, m \to \infty} p(x_n, x_m) = 0 \quad (20)$$
We assert that $Tz = z$. Due to (18) and (PM4), we have

$$p(Tz, z) \leq p(Tz, Tx_n) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1})$$

$$\leq p(z, x_n) - \phi(p(z, x_n)) + p(x_{n+1}, z) - p(x_{n+1}, x_{n+1}) \tag{21}$$

Letting $n \to \infty$ and regarding the continuity of $\phi$, then (16) yields that $p(Tz, z) \leq 0$. Hence $Tz = z$. □

If we take $\Phi(t) = t - \phi(t)$, then one can easily see that $\Phi$ satisfies all conditions of the main theorem of [6]. So we can state some results of [6] as a corollary of our theorem.

**Corollary 2.4** (See [6]) Let $(X, \preceq)$ be a partially ordered set and $(X, p)$ a complete partial metric space. Suppose $T : X \to X$ be a self mapping such that

$$p(Tx, Ty) \leq \Phi(p(x, y)), \text{ for all } x, y \in X, \text{ with } x \preceq y$$

where $\Phi(t) : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing function such that $\phi(t) < t$ for each $t > 0$. Then $T$ has a unique fixed point.

If we take $\Phi(t) = kt$ we get Banach contraction principle for PMS.

**Corollary 2.5** (See [2, 4, 6]) Let $(X, \preceq)$ be a partially ordered set and $(X, p)$ a complete partial metric space. Suppose $T : X \to X$ be a self mapping such that

$$p(Tx, Ty) \leq kp(x, y), \text{ for all } x, y \in X, \text{ with } x \preceq y$$

where $k \in [0, 1)$. Then $T$ has a unique fixed point.

**Example 2.6** Let $X = \mathbb{R}^+$ and $p(x, y) = \max\{x, y\}$ then $(X, p)$ is a PMS (See e.g. [6]). Suppose $T : X \to X$ such that $Tx = \begin{cases} \frac{x^2}{1+x} & \text{for all } x \in [0, 1] \\ \frac{2x}{1+x} & \text{for all } x \in (1, \infty) \end{cases}$ and $\phi(t) : [0, \infty) \to [0, \infty)$ such that $\phi(t) = \frac{t}{1+t}$. It is clear that $T$ is non-decreasing. For $x < y$ we have

$$p(Tx, Ty) = \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} = \frac{x^2}{1+x} \leq x - \frac{x}{1+x} = \frac{x^2}{1+x}$$

Thus, it satisfies all conditions of the Theorem 2.3. Notice also that, for choosing $\Phi(t) = t - \phi(t) = \frac{t^2}{1+t}$, all conditions of Theorem 1 of [6] and guarantee that $T$ has a unique fixed point, indeed $x = 0$ is the required point.
References


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