Mathematica Aeterna, Vol. 3, 2013, no. 8, 645 - 653

Totally umbilical Hemi-slant submanifolds of Cosymplectic manifolds

Meraj Ali Khan

Department of Mathematics, Faculty of Science

University of Tabuk, Saudi Arabia

Abstract

In the present paper we have study totally umbilical hemi-slant submanifolds of Cosymplectic manifolds via Riemannian curvature tensor and finally obtained a classification for the Totally umbilical hemi-slant submanifolds of Cosymplectic manifolds.

Mathematics Subject Classification: 53C25, 53C40, 53C42, 53D15.

Keywords: Totally umbilical, hemi-slant submanifolds, Cosymplectic manifolds.

1 Introduction

The study of slant submanifolds was initiated by B. Y. Chen [3]. Since then many research articles have been appeared in this field, slant submanifolds are the natural generalization of both holomorphic and totally real submanifolds. A. Lotta [2] defined and studied these submanifolds in the setting of contact manifolds. Later on, J. L. Caberizo et al. [6, 7] studied slant, semi-slant and bi-slant submanifolds in contact geometry . In particular, totally umbilical proper slant submanifolds of Kaehler manifolds has been studied in [5].

The idea of hemi-slant submanifolds was introduced by A. Carriazo as a particular class of bi-slant submanifolds and he called them anti-slant submanifold after that, V.A. Khan and M. A. Khan [10] named these submanifolds Pseudo-slant submanifolds and studied them in the setting of Sasakian manifold. Recently, these submanifolds studied by B. Sahin for their warped product [6]. In this paper we will study hemi-slant submanifolds of Cosymplectic manifolds.

2 Preliminary Notes

A $2n+1-$dimensional C^∞-manifold $\bar{M}$ is called A $2n+1$ dimensional C^∞ manifold $\bar{M}$ is said to have an almost contact structure if there exist on M a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η satisfying.

$$\phi^2 = -I + \eta \otimes \xi, \;\; \phi(\xi) = 0, \;\; \eta \circ \phi = 0, \;\; \eta(\xi) = 1. \tag{1}$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying following conditions

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \;\; \eta(X) = g(X, \xi) \tag{2}$$

where X, Y are vector fields on $\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

where f is the $C^\infty-$function on $\bar{M} \times R$. The condition for normality in terms of ϕ, ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\bar{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Finally the fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be cosymplectic, if it is normal and both Φ and η are closed, and structure equation of cosymplectic manifold is given by

$$(\bar{\nabla}_X \phi)Y = 0 \tag{3}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of $\bar{M}$ and $\bar{\nabla}$ denotes the Riemannian connection of the metric g. Moreover for cosymplectic manifold

$$\bar{\nabla}_X \xi = 0. \tag{4}$$

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connection on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{6}$$

for each $X, Y \in TM$ and $V \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$ and they are related as

$$g(h(X,Y), N) = g(A_N X, Y), \tag{7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

For any $X \in TM$, we write

$$\phi X = TX + FX, \tag{8}$$

where TX is the tangential component and FX is the normal component of ϕX.

Similarly, for any $V \in T^{\perp}M$, we write

$$\phi V = tV + fV, \tag{9}$$

where tV is the tangential component and fV is the normal component of ϕV. The covariant derivatives of the tensor field T and F are defined as

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{11}$$

From equations (3)(5), (6), (8) and (9) we have

$$(\bar{\nabla}_X T)Y = A_{FY}X + th(X,Y) \tag{12}$$

$$(\bar{\nabla}_X F)Y = fh(X,Y) - h(X,TY). \tag{13}$$

The mean curvature vector H on M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_j)$$

where n is the dimension of M and $\{e_1, e_2, ...e_n\}$ is the local orthonormal frame of vector fields on M.

A submanifold M of Riemannian manifold $\bar{M}$ is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \tag{14}$$

If $h(X,Y)=0$ for any $X,Y\in TM$ then M is said to be totally geodesic. If $H=0$, then it is said to be minimal.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant submanifold if for any $x\in M$ and $X\in T_xM$ is constant. The constant angle $\theta\in[0,\pi/2]$ is then called slant angle of M in $\bar{M}$. If $\theta=0$ the submanifold is invariant submanifold, if $\theta=\pi/2$ then it is anti-invariant submanifold if $\theta\neq 0,\pi/2$ then it is proper slant submanifold.

For slant submanifolds of contact manifolds J. L. Cabrerizo et al. [6] proved the following Lemma

Lemma 2.1 *Let M be a submanifold of an almost contact manifold $\bar{M}$, such that $\xi\in TM$ then M is slant submanifold if and only if there exist a constant $\lambda\in[0,1]$ such that*

$$T^2=\lambda(I-\eta\otimes\xi). \tag{15}$$

Thus, one has the following consequences of above formulae

$$g(TX,TY)=\cos^2\theta[g(X,Y)-\eta(X)\eta(Y)]$$

$$g(FX,FY)=\sin^2\theta[g(X,Y)-\eta(X)\eta(Y)]$$

Definition 2.2 *A submanifold M of $\bar{M}$ is said to be hemi-slant submanifold of an almost contact manifold $\bar{M}$ if there exist two orthogonal complementary distribution D_1 and D_2 on M such that*

(i) $TM=D_1\oplus D_2\oplus\langle\xi\rangle$.

(ii) The distribution D_1 is anti-invariant i.e., $\phi D_1\subseteq T^{\perp}M$.

(iii) The distribution D_2 is slant with slant angle $\theta\neq\pi/2$.

If μ is invariant subspace under ϕ of the normal bundle $T^{\perp}M$, then in the case of hemi-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M=\mu\oplus\phi D^{\perp}\oplus FD_{\theta}.$$

The Riemannian curvature tensor is defined as

$$\bar{R}(X,Y)Z=\bar{\nabla}_X\bar{\nabla}_YZ-\bar{\nabla}_Y\bar{\nabla}_XZ-\bar{\nabla}_{[X,Y]}Z \tag{16}$$

The equation of Coddazi for totally umbilical hemi-slant submanifold M is given by

$$\bar{R}(X,Y,Z,V)=g(Y,Z)g(\nabla^{\perp}_XH,V)-g(X,Z)g(\nabla^{\perp}_YH,V) \tag{17}$$

where $\bar{R}(X,Y,Z,V) = g(\bar{R}(X,Y)Z,V)$ and X,Y,Z are vector fields on M and $V \in T^{\perp}M$.

It is easy to see that Riemannian curvature tensor for Cosymplectic manifold satisfies the following properties

$$(a) \ \ \bar{R}(\phi X, \phi Y)Z = \bar{R}(X,Y)Z \ \ (b) \ \ \phi\bar{R}(X,Y)Z = \bar{R}(X,Y)\phi Z. \tag{18}$$

By an extrinsic sphere we mean a submanifold of an arbitrary Riemannian manifold which is totally umbilical and has a nonzero parallel mean curvature vector [9].

3 Main Results

In this section, we will study a special class of hemi-slant submanifolds which are totally umbilical. Throughout the section we consider M as a totally umbilical hemi-slant submanifold of a Cosymplectic manifold. Now we have the following theorem

Theorem 3.1 *Let M be a totally umbilical hemi-slant submanifold of a Cosymplectic manifold $\bar{M}$ such that the mean curvature vector $H \in \mu$. Then one of the following statement is true*

(i) M is totally geodesic.

(ii) M is semi-invariant submanifold.

Proof. For $V \in \phi D^{\perp}$ and $X \in D_{\theta}$, we have

$$\bar{\nabla}_X \phi V = \phi \bar{\nabla}_X V \tag{19}$$

using equations (5),(6) and (15) the above equation becomes

$$\nabla_X \phi V + g(X, \phi V)H = -\phi X g(X,V) + \phi \nabla^{\perp}_X V.$$

Then by orthogonality of two distributions and the assumption $H \in \mu$ the above equation takes the form

$$\nabla_X \phi V = \phi \nabla^{\perp}_X V \tag{20}$$

which implies that $\nabla^{\perp}_X V \in \phi D^{\perp}$, for any $V \in \phi D^{\perp}$. Also we have $g(V,H) = 0$, for $V \in FD^{\perp}$, then using this fact we derive

$$g(\nabla^{\perp}_X V, H) = -g(V, \nabla^{\perp}_X H) = 0. \tag{21}$$

The above equation implies

$$\nabla^{\perp}_X H \in \mu \oplus FD_\theta.$$

Now, for any $X \in D_\theta$, we have

$$\bar{\nabla}_X \phi H = \phi \bar{\nabla}_X H,$$

using equation (15), we obtain

$$A_{\phi H} X + \nabla^{\perp}_X \phi H = -\phi A_H X + \phi \nabla^{\perp}_X H.$$

Now, using the assumption, that M is totally umbilical the above equation takes the form

$$-Xg(H, \phi H) + \nabla^{\perp}_X \phi H = -\phi X g(H, H) + \phi \nabla^{\perp}_X H,$$

using the equation (4) above equation takes the form

$$\nabla^{\perp}_X \phi H = -TXg(H, H) - FXg(H, H) + \phi \nabla^{\perp}_X H,$$

taking Inner product with $FX \in FD_\theta$ and using the equation (15)

$$g(\nabla_X \phi H, FX) = -\sin^2 \theta \|H\|^2 \|X\|^2 + g(\phi \nabla^{\perp}_X H, FX).$$

Then from equation (15), the last term of right hand side is identically zero, thus the above equation becomes

$$g(\nabla_X \phi H, FX) + \sin^2 \theta \|H\|^2 \|X\|^2 = 0. \tag{22}$$

Since equation (22) has a solution if either $H \neq 0$, then $D_\theta = \{0\}$ i.e., M is totally real submanifold and if $D_\theta \neq \{0\}$ then M is totally geodesic submanifold or M is semi-invariant submanifold.

Now for any $Z \in D^{\perp}$, by equation (12)

$$-T\nabla_Z Z = A_{\phi Z} Z + th(Z, Z).$$

Taking Inner product with $W \in D^{\perp}$ the above equation takes the form

$$-g(T\nabla_Z Z, W) = g(A_{FZ} Z, W) + g(th(Z, Z), W).$$

As M is totally umbilical hemi-slant submanifold, then above equation becomes

$$g(Z, W)g(H, FZ) + g(tH, W)\|Z\|^2 = 0. \tag{23}$$

The above equation has a solution if either $H \in \mu$ or $\dim D^{\perp} = 1$.

If moreover, $H \notin \mu$ then

Theorem 3.2 *Let M be a totally umbilical hemi-slant submanifold of a Cosymplectic manifold $\bar{M}$ such that dimension of slant distribution $D_\theta \geq 4$ and F is parallel, then M is either*

(i) extrinsic sphere.

(ii) or anti-invariant submanifold.

Proof. Since dimension of slant distribution $D_\theta \geq 4$, then we can choose a set of orthogonal vectors $X, Y \in D_\theta$, such that $g(X, Y) = 0$. Now from equation (18)(b), we have

$$\phi\bar{R}(X, Y)Z = \bar{R}(X, Y)\phi Z$$

for any $X, Y, Z \in D_\theta$. Replacing Z by TY, we obtain

$$\phi\bar{R}(X, Y)TY = \bar{R}(X, Y)\phi TY.$$

Using equations (8) and (1), the above equation takes the form

$$\phi\bar{R}(X, Y)TY = -\cos^2\theta\bar{R}(X, Y)Y + \bar{R}(X, Y)FTY. \tag{24}$$

On the other hand, since F is parallel, then we have

$$\bar{R}(X, Y)FTY = F\bar{R}(X, Y)TY. \tag{25}$$

Then by equations (24) and (25) we have

$$\phi\bar{R}(X, Y)TY = -\cos^2\theta\bar{R}(X, Y)Y + F\bar{R}(X, Y)TY. \tag{26}$$

Taking Inner product in equation (26) by $N \in T^\perp M$, we get

$$g(\phi\bar{R}(X, Y)TY, N) = -\cos^2\theta g(\bar{R}(X, Y)Y, N) + g(F\bar{R}(X, Y)TY, N),$$

using equation (8) the above equation reduced to

$$\cos^2\theta g(\bar{R}(X, Y, Y, N) = 0. \tag{27}$$

Then, from equation (17), we derive

$$\cos^2\theta g(Y, Y)g(\nabla^\perp_X H, N) - g(X, Y)g(\nabla^\perp_Y H, N) = 0.$$

Since X and Y are orthogonal vectors, then the above equation gives

$$\cos^2\theta g(\nabla^\perp_X H, N)\|Y\|^2 = 0. \tag{28}$$

The equation (28) has a solution either $\theta = \pi/2$ i.e., M is anti-invariant or $\nabla^\perp_X H = 0\ \forall X \in D_\theta$. By similar calculation for any $X \in D^\perp \oplus < \xi >$ we can obtain $\nabla^\perp_X H = 0$, hence $\nabla^\perp_X H = 0$ for all $X \in TM$ i.e., the mean curvature vector H is parallel to submanifold, i.e., M is extrinsic sphere.

Now we are in position to prove our main theorem

Theorem 3.3 *Let M be a totally umbilical hemi-slant submanifold of a Cosymplectic manifold $\bar{M}$. Then M is either*

(i) Totally geodesic,

(ii) or Semi-invariant,

(iii) or dim $D^{\perp} = 1$,

(iv) or Extrinsic sphere.

case (iv) holds if F is parallel and dim $M \geq 5$(odd)

Proof. If $H \in \mu$ then by Theorem 3.1 M is either totally geodesic or semi-invariant submanifolds which are case (i) and (ii). If $H \notin \mu$, then equation (24) has a solution if dim $D^{\perp} = 1$ which is case (iii) and moreover if $H \notin \mu$ and F is parallel on M then by Theorem 3.2 M is extrinsic sphere which proves the theorem completely.

References

[1] A. Carriazo, New Developments in Slant Submanifolds Theory, Narosa Publishing House, New Delhi, India, 2002.

[2] A. Lotta, Slant Submanifolds in Contact Geometry, Bull. Math. Soc. Roum., 39 (1996), 183-198.

[3] B.Y. Chen, Slant Immersions, Bull. Aust. Math. Soc., 41 (1990), 135-147.

[4] B. Sahin, Warped product submanifolds of Kaehler manifolds with a slant factor Ann. Pol. Math. 95 (2009), 207-226.

[5] B. Sahin, Every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic, Result. Math. 54 (2009), 167-172.

[6] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez, and M. Fernandez, Slant Submanifolds in Sasakian Manifolds, Glasgow Math. J., 42 (2000), 125-138.

[7] J.L. Cabrerizo, A. Carriazo, L.M. Fernandez, and M. Fernandez, Semi-slant Submanifolds in Sasakian Manifolds, Geometry Dedicata, 78 (1999), 183-199.

[8] N. Papaghiuc, Semi-slant Submanifolds of Kaehlerian Manifold, An. Stiint. Univ. Iasi, 9 (f_1) (1994), 55-61.

[9] S. Deshmukh and S. I. Husain,Totally umbilical CR-submanifolds of a Kaehler manifold, Kodai Math. J. 9(3) (1986), 425-429.

[10] V. A. Khan and M. A. Khan Pseudo-slant submanifolds of a Sasakian manifold, Indian J. Pure Appl. Math., 38(2007) 31-42.

Received: September, 2013