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Three classes of 3-Lie algebras

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Abstract

This paper studies three classes of 3-Lie algebras which are realized by bilinear functions on vector spaces. The solvability, nilpotency and metric structures of 3-Lie algebras are discussed. And structures of inner derivation algebras and derivation algebras are investigated. The results can be used in the realization of 3-Lie algebras.

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1 Introduction

3-Lie algebras [1] are very important ternary algebraic system since the wide applications in many fields on mathematics, mathematical physics and string theory (cf. [2, 3]). In the papers [4, 5], the 3-Lie algebras are realized by Lie algebras, linear functions and cubic matrices. And in paper [6], three classes of 3-Lie algebras $(V, [\,,\,,\,]_{f,\lambda})$ are constructed by bilinear functions f on a vector spaces V. In this paper we study the solvability, nilpotency and metric structures of the 3-Lie algebras $(V, [\,,\,,\,]_{f,\lambda})$, and their derivation algebras.

In this paper we suppose that F is a field of characteristic zero. And the multiplications of basis vectors which are not listed in the multiplication table are assumed to be zero.

A 3-Lie algebra is a vector space L over a field F endowed with a 3-ary multi-linear skew-symmetric operation $[x_1, x_2, x_3]$ satisfying the 3-Jacobi identity

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3} [x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3], \ \forall x_1, x_2, x_3 \in L. \quad (1)$$

A derivation of a 3-Lie algebra L is a linear map $D : L \to L$, such that for any elements x_1, x_2, x_3 of L

$$D([x_1, x_2, x_3]) = \sum_{i=1}^{3} [x_1, \ \cdots, \ D(x_i), \ \cdots, \ x_3]. \quad (2)$$

The set of all derivations of L is a subalgebra of Lie algebra $gl(L)$. This subalgebra is called *the derivation algebra* of A, and is denoted by $Der(L)$. The map $ad(x_1, x_2) : \ L \to \ L$ defined by $ad(x_1, x_2)(x) = [x_1, x_2, x]$ for $x_1, x_2, \ x \in L$ is called *a left multiplication*. It follows from (2) that $ad(x_1, x_2)$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $Der(L)$ and is denoted by $ad(L)$. Every element in $ad(L)$ is by definition *an inner derivation*, and for $\forall \ x_1, x_2, y_1, y_2$ of L,

$$[ad(x_1, x_2), \mathrm{ad}(y_1, y_2)] = ad([x_1, x_2, y_1], y_2) + ad(y_1, [x_1, x_2, y_2]).$$

A metric on a 3-Lie algebra L is a non-degenerate symmetric bilinear form $B : L \times L \to F$ satisfying

$$B([x, y, z], u) + B(z, [x, y, u]) = 0, \forall x, y, z, u \in L. \quad (3)$$

Lemma 1.1 [6] *Let V be a linear space over a field F with $dimV = n \geq 6$, c be a fixed nonzero vector of V, $f, g, h : V \otimes V \to F$, $\lambda : V \otimes V \otimes V \longrightarrow F$. Define the 3-ary multiplication on V as follows: for arbitrary $x, y, z \in V$,*

$$[x, y, z]_{f,\lambda} = f(y, z)x + g(z, x)y + h(x, y)z + \lambda(x, y, z)c. \quad (4)$$

Then $(V, [, ,]_{f,\lambda})$ is a 3-Lie algebra if and only if

(1) $c \in Kerf$, and $f = g = h$, f is a bilinear skew-symmetric form on V satisfying $f(x_2, x_3)x_1 + f(x_3, x_1)x_2 + f(x_1, x_2)x_3 \in Kerf, \forall x_1, x_2, x_3 \in V$.

(2) λ is a ternary linear skew-symmetric function on V and for arbitrary $x_1, x_2, x_3, y_2, y_3 \in V$, λ satisfies

$\lambda([x_1, x_2, x_3]_{f+\lambda_c}, y_2, y_3) - \lambda(x_1, x_2, x_3)(f + \lambda_c)(y_2, y_3)$
$= \lambda([x_1, x_2, x_3]_f, y_2, y_3) - \lambda([x_1, y_2, y_3]_f, x_2, x_3) - \lambda(x_1, [x_2, y_2, y_3]_f, x_3)$
$-\lambda(x_1, x_2, [x_3, y_2, y_3]_f)$,

where $\lambda_c(x, y) = \lambda(c, x, y)$, $[x_1, x_2, x_3]_f$ and $[x_1, x_2, x_3]_{f+\lambda_c}$ are defined as

$[x, y, z]_f = f(y, z)x + g(z, x)y + h(x, y)z.$

Lemma 1.2 [6] *Let* $(V, [, ,]_{f,\lambda})$ *be a* 3*-Lie algebra in Lemma 1.1 with a basis* $\{z_1, \cdots, z_n\}$, $n \geq 6$. *If* λ *satisfies*
$= \lambda([x_1, x_2, x_3]_f, y_2, y_3)$
$= \lambda([x_1, y_2, y_3]_f, x_2, x_3) + \lambda(x_1, [x_2, y_2, y_3]_f, x_3) + \lambda(x_1, x_2, [x_3, y_2, y_3]_f)$
then $(V, [, ,]_{f,\lambda})$ *is isomorphic to one and only one of the following: for every* $\alpha \in F, \alpha \neq 0$,

$(a)[z_1, z_2, z_3]_{f,\lambda} = \alpha z_3, [z_1, z_2, z_i]_{f,\lambda} = z_i, \quad 3 < i \leq n;$

$(b)[z_1, z_2, z_3]_{f,\lambda} = \alpha z_3, \quad [z_1, z_2, z_4]_{f,\lambda} = z_4 + z_3, \quad [z_1, z_2, z_j]_{f,\lambda} = z_j, 5 \leq j \leq n;$

$(c)[z_1, z_2, z_3]_{f,\lambda} = 0, [z_1, z_2, z_i]_{f,\lambda} = z_i, \ 3 < i \leq n.$

2 Structures on 3-Lie Algebras $(V, [, ,]_{f,\lambda})$

In this section we first discuss the metric structures on the 3-Lie algebras $(V, [, ,]_{f,\lambda})$. For the simplicity, in the following the multiplication $[, ,]_{f,\lambda}$ is denoted by $[, ,]$.

Theorem 2.1 *There does not exist metric structures on the 3-Lie algebras in Lemma 1.2.*

Proof. Let $B : V \times V \to F$ be a bilinear symmetric form on V which satisfies Eq (4). Then by Lemma 1.2, if $(V, [, ,]_{f,\lambda})$ is a 3-Lie algebra of the case (a), we have

$B(z_3, z_1) = B([z_1, z_2, z_3], z_1) = B(z_2, [z_3, z_1, z_1]) = 0,$
$B(z_3, z_2) = B([z_3, z_1, z_2], z_2) = B(z_3, [z_1, z_2, z_2]) = 0,$
$B(z_3, z_3) = B([z_1, z_2, z_3], z_3) = B(z_1, [z_2, z_3, z_3]) = 0,$
$B(z_3, z_j) = B(z_3, [z_1, z_2, z_j]) = B(z_1, [z_2, z_3, z_j]) = 0, \ 4 \leq j \leq m,$

Therefore, $B(z_3, V) = 0$, that is, B is degenerated. Therefore, there does not exist metric structures on the 3-Lie algebra of the case (a).

By the similar discussion, there do not exist metric structures on the 3-Lie algebras of the case (b) and (c). The proof is completed.

Theorem 2.2 *The* 3*-Lie algebras* $(V, [, ,]_{f,\lambda})$ *in Lemma 1.2 are two-step solvable, but non-nilpotent.*

Proof. By Lemma 1.2, for the 3-Lie algebras of the cases (a) and (b), we have

$$V^{(1)} = [V, V, V] = \sum_{i=3}^{n} F z_i, V^{(2)} = [V^{(1)}, V^{(1)}, V] = [\sum_{i=3}^{n} F z_i, \sum_{i=3}^{n} F z_i, V] = 0.$$

In the case (c),

$$V^{(1)} = [V, V, V] = \sum_{i=4}^{n} F z_i, V^{(2)} = [V^{(1)}, V^{(1)}, V] = [\sum_{i=4}^{n} F z_i, \sum_{i=4}^{n} F z_i, V] = 0.$$

Therefore, the 3-Lie algebras are two-step solvable.

Since the left multiplication $ad(z_1, z_2)$ has the eigenvalue 1, $ad(z_1, z_2)$ is non-nilpotent. Therefore, the 3-Lie algebras in Lemma 1.2 are non-nilpotent. The proof is completed.

In the following we study the derivation algebras of 3-Lie algebras in Lemma 1.2. Let $D : V \rightarrow V$ be any derivation of V, and let the matrix form of D in the basis $\{z_1, \cdots, z_n\}$ be $A = (a_{ij})$, $1 \leq i, j \leq n$, that is, $D(z_i) = \sum\limits_{j=1}^{n} a_{ij} z_j$. Then $D = \sum\limits_{i,j=1}^{n} a_{ij} E_{ij}$, where E_{ij} is the matrix unit with the number 1 in the position i^{th}-row and j^{th}-column, $1 \leq i, j \leq n$.

Theorem 2.3 *For 3-Lie algebras in Lemma 1.2, we have the following result:*

1) If $(V, [,,]_{f,\lambda})$ is the case (a), then $\dim ad(V) = 2n - 3$ and

$$ad(V) = F\Big(\alpha E_{33} + \sum_{k=4}^{n} E_{kk}\Big) + \sum_{k=3}^{n} \Big(FE_{1k} + FE_{2k}\Big). \tag{5}$$

2) If $(V, [,,]_{f,\lambda})$ is the case (b), then $\dim ad(V) = 2n - 3$ and

$$ad(V) = F\Big(\alpha E_{33} + E_{43} + \sum_{k=3}^{n} E_{kk}\Big) + \sum_{k=3}^{n} \Big(FE_{1k} + FE_{2k}\Big). \tag{6}$$

3) If $(V, [,,]_{f,\lambda})$ is the case (c), then $\dim ad(V) = 2n - 5$ and

$$ad(V) = F\Big(\sum_{k=4}^{n} E_{kk}\Big) + \sum_{k=4}^{n} \Big(FE_{1k} + FE_{2k}\Big). \tag{7}$$

Proof. If $(V, [,,]_{f,\lambda})$ is the case (a), by the direct computation by Lemma 1.2, the matrix form of left multiplications $ad(z_i, z_j)$ are as follows:

$$ad(z_1, z_2) = \alpha E_{33} + \sum_{k=4}^{n} E_{kk}, ad(z_1, z_3) = -\alpha E_{23}, ad(z_1, z_k) = -E_{2k}, 4 \leq k \leq n,$$

$$ad(z_2, z_3) = \alpha E_{13}, ad(z_2, z_k) = E_{1k}, 4 \leq k \leq n.$$

Therefore, $\{ad(z_1, z_k), ad(z_2, z_l), 2 \leq k \leq n, 3 \leq l \leq n\}$ is a basis of $ad(V)$, we obtain Eq.(5) and $\dim ad(V) = 2n - 3$.

If $(V, [,,]_{f,\lambda})$ is the case (a), by Lemma 1.2, the matrix form of left multiplications $ad(z_i, z_j)$ are as follows:

$$ad(z_1, z_2) = \alpha E_{33} + E_{43} + \sum_{k=4}^{n} E_{kk}, ad(z_1, z_3) = -\alpha E_{23}, ad(z_1, z_4) = -E_{24} - E_{23},$$

$$ad(z_1, z_k) = -E_{2k}, 5 \leq k \leq n, ad(z_2, z_3) = \alpha E_{13}, ad(z_2, z_4) = E_{14} + E_{13},$$

$$ad(z_2, z_k) = E_{1k}, 5 \le k \le n.$$

Therefore, $\{ad(z_1, z_k), ad(z_2, z_l), 2 \le k \le n, 3 \le l \le n\}$ is a basis of $ad(V)$, we obtain Eq.(6) and $\dim ad(V) = 2n - 3$.

f $(V, [,,]_{f,\lambda})$ is the case (c), by Lemma 1.2,

$$ad(z_1, z_2) = \sum_{k=4}^{n} E_{kk}, ad(z_1, z_k) = -E_{2k}, ad(z_2, z_k) = E_{1k}, 4 \le k \le n.$$

Therefore, $\{ad(z_1, z_k), ad(z_2, z_j), 2 \le k \le n, 3 \neq k, 4 \le j \le n\}$ is a basis of $ad(V)$, we obtain Eq.(7) and $\dim ad(V) = 2n - 5$. The proof is completed.

Theorem 2.4 *For 3-Lie algebras in Lemma 1.2, the derivation algebras are as follows:*

1) For the case (a)*, if* $\alpha = 1$, $\dim Der(V) = n^2 - 2n + 3$,

$Der(V) = F(E_{11} - E_{22}) + \sum_{k=2}^{n} FE_{1k} + \sum_{k\neq 2,k=1}^{n} FE_{2k} + \sum_{j,k=3}^{n} FE_{jk}$.

If $\alpha \neq 1$, $\dim Der(V) = n^2 - 4n + 8$,

$Der(V) = F(E_{11} - E_{22}) + \sum_{k=2}^{n} FE_{1k} + \sum_{k\neq 2,k=1}^{n} FE_{2k} + \sum_{j,k=4}^{n} FE_{jk}$.

2) For the case (b), $\dim Der(V) = n^2 - 5n + 13$,

$Der(V) = F(E_{11} - E_{22}) + \sum_{j=2}^{n} FE_{1j} + \sum_{j\neq 2,j=1}^{n} FE_{2j} + F(E_{33} + E_{44})$

$+F(E_{43} + (\alpha - 1)E_{44}) + \sum_{j=5}^{n} F(E_{j3} + (1 - \alpha)E_{j4}) + \sum_{j,k=5}^{n} FE_{jk}$.

3) For the case (c), $\dim Der(V) = n^2 - 4n + 11$,

$Der(V) = F(E_{11} - E_{22}) + \sum_{k=2}^{n} FE_{1k} + \sum_{k\neq 2,k=1}^{n} FE_{2k} + \sum_{k=1}^{3} FE_{3k} + \sum_{j,k=4}^{n} FE_{jk}$.

Proof. If $(V, [,,]_{f,\lambda})$ is the case (a), by Lemma 1.2 and $D([z_1, z_2, z_3]) = [D(z_1), z_2, z_3] + [z_1, D(z_2), z_3] + [z_1, z_2, D(z_3)] = \alpha D(z_3)$, we have

$$\alpha \sum_{k=1}^{n} a_{3k} z_k = \alpha(a_{11} + a_{22} + a_{33})z_3 + \sum_{k=4}^{n} a_{3k} z_k,$$

then we have

$a_{31} = a_{32} = a_{11} + a_{22} = 0, \alpha a_{3k} = a_{3k}, 4 \le k \le n.$

From $D([z_1, z_2, z_j]) = [D(z_1), z_2, z_j] + [z_1, D(z_2), z_j] + [z_1, z_2, D(z_j)] = D(z_j)$, for $4 \le j \le n$, we have

$\sum_{k=1}^{n} a_{jk} z_k = (a_{11} + a_{22} + \alpha a_{j3})z_3 + \sum_{k=4}^{n} a_{jk} z_k,$

then we have $a_{j1} = a_{j2} = 0, a_{11} + a_{22} + (\alpha - 1)a_{j3} = 0, 4 \le j \le n$.

Summarizing above discussions, we have the matrix form of D is in the case $\alpha = 1$,

$D = a_{11}(E_{11} - E_{22}) + a_{12}E_{12} + a_{21}E_{21} + \sum_{k=3}^{n} (a_{1k}E_{1k} + a_{2k}E_{2k}) + \sum_{j,k=3}^{n} a_{jk}E_{jk}$.

In the case $\alpha \neq 1$, $D = a_{11}(E_{11} - E_{22}) + a_{12}E_{12} + a_{21}E_{21} + \sum\limits_{k=3}^{n}(a_{1k}E_{1k} + a_{2k}E_{2k}) + \sum\limits_{j,k=4}^{n} a_{jk}E_{jk}$. The result 1) is follows.

If $(V, [,,]_{f,\lambda})$ is the case (b), from $D([z_1, z_2, z_3]) = \alpha D(z_3)$, we have

$$\alpha \sum_{k=1}^{n} a_{3k}z_k = (\alpha a_{11} + \alpha a_{22} + \alpha a_{33} + a_{34})z_3 + \sum_{k=4}^{n} a_{3k}z_k,$$

then $a_{31} = a_{32} = \alpha(a_{11} + a_{22}) + a_{34} = 0, \alpha a_{3k} = a_{3k}, 4 \leq k \leq n$.

Since $D([z_1, z_2, z_4]) = D(z_3 + z_4)$, we get

$$(a_{11} + a_{22} + a_{44})(z_4 + z_3) + \alpha a_{43}z_3 + \sum_{k=5}^{n} a_{4k}z_k = \sum_{k=1}^{n}(a_{4k} + a_{3k})z_k.$$

Then we have $a_{41} = a_{42} = 0$, $a_{11} + a_{22} + a_{44} = a_{44} + a_{34}$, $a_{3k} = 0, 5 \leq k \leq n$, $a_{11} + a_{22} + a_{44} + \alpha a_{43} = a_{43} + a_{33}$.

From $D([z_1, z_2, z_j]) = D(z_j)$, for $5 \leq j \leq n$, we have

$$\sum_{k=1}^{n} a_{jk}z_k = (a_{11} + a_{22})z_j + (\alpha a_{j3} + a_{j4})z_3 + \sum_{k=4}^{n} a_{jk}z_k.$$

Summarizing above discussions, we have

$$a_{11} + a_{22} = 0, a_{j1} = a_{j2} = a_{3k} = 0, 4 \leq k \leq n, 3 \leq j \leq n;$$

$$a_{44} + (\alpha - 1)a_{43} = a_{33}, a_{j4} = (1 - \alpha)a_{j3}, 5 \leq j \leq n.$$

$D = a_{11}(E_{11} - E_{22}) + \sum\limits_{j=2}^{n} a_{1j}E_{1j} + \sum\limits_{j\neq 2, j=1}^{n} a_{2j}E_{2j} + a_{33}(E_{33} + E_{44})$
$+ a_{43}(E_{43} + (\alpha - 1)E_{44}) + \sum\limits_{j=5}^{n} a_{j3}(E_{j3} + (1 - \alpha)E_{j4}) + \sum\limits_{j,k=5}^{n} a_{jk}E_{jk}$.

The result 2) is follows.

By the completely similar discussions to above, for the case (c),

$D = a_{11}(E_{11} - E_{22}) + \sum\limits_{k=2}^{n} a_{1k}E_{1k} + \sum\limits_{k\neq 2, k=1}^{n} a_{2k}E_{2k} + \sum\limits_{k=1}^{3} a_{3k}E_{3k} + \sum\limits_{j,k=4}^{n} a_{jk}E_{jk}$.

The proof is completed.

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