The quasiderivations of Hom-Lie color algebras

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Abstract

In this paper, we give some basic properties of a Hom-Lie color algebra $L$. In particular, we prove that the quasiderivations of $L$ can be embedded as derivations in a larger Hom-Lie color algebra, and obtain a direct sum decomposition of $\text{Der}(L)$ when the annihilator of $L$ is equal to zero.

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1 Introduction

Hom-Lie algebras are a generalization of Lie algebras, Hom-Lie algebras are also related to deformed vector fields, the various versions of the Yang-Baxter equations, braid group representations, and quantum groups [5]. More applications of the Hom-Lie algebras, Hom-algebras can be found in [4, 6]. The purpose of this paper is to generalize some beautiful results to the Quasiderivations of a Hom-Lie color algebra.

2 Preliminary Notes

Throughout this paper $K$ is a field, a vector space $V$ is $\Gamma$-graded.

Definition 2.1 [1] Let $K$ and $\Gamma$ be an abelian group, a map $\Gamma \times \Gamma \to K^*$ is called a skew-symmetric bi-character on $\Gamma$ if the following identities hold for all $x,y,z$ in $\Gamma$

(1) $\varepsilon(x,y)\varepsilon(y,x) = 1,$

(2) $\varepsilon(x,y+z) = \varepsilon(x,y)\varepsilon(x,z),$
The following subvector space \( \mathfrak{U} \) of \( \text{End}(L) \) consisting of even linear maps \( u \) on \( L \) as follows:
\[
\mathfrak{U} = \{ u \in \text{End}(L) \mid u\alpha = \alpha u \}
\]
and \( \sigma : \mathfrak{U} \to \mathfrak{U}; \sigma(u) = \alpha u \). Then \( \mathfrak{U} \) is a Hom-Lie color algebra over \( K \) with the bracket
\[
[D_\theta, D_\mu] = D_\theta D_\mu - \varepsilon(\theta, \mu) D_\mu D_\theta
\]
for all \( D_\theta, D_\mu \in \text{hg} \mathfrak{U} \).

**Definition 2.4** [3] Let \( (L, [\cdot, \cdot], \alpha) \) be a multiplicative Hom-Lie color algebra. A homogeneous bilinear map \( D : L \to L \) is said to be an \( \alpha^k \)-derivation of \( L \), where \( k \in \mathbb{N} \), if it satisfies
\[
D\alpha = \alpha D,
\]
\[
[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D(y)] = D([x, y]),
\]
\( \forall x \in \text{hg}(L), y \in L \).

We denote the set of all \( \alpha^k \)-derivations by \( \text{Der}_{\alpha^k}(L) \), then \( \text{Der}(L) := \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(L) \) provided with the super-commutator and the following even map
\[
\tilde{\alpha} : \text{Der}(L) \to \text{Der}(L); \quad \tilde{\alpha}(D) = D\alpha
\]
is a Hom-subalgebra of \( \mathfrak{U} \) and is called the derivation algebra of \( L \).

**Definition 2.5** [1] An endomorphism \( D \in \text{hg}(\text{Der}(L)) \) is said to be a homogeneous \( \alpha^k \)-quasiderivation, if there exists an endomorphism \( D' \in \text{hg}(\text{End}(L)) \) such that
\[
D\alpha = \alpha D, D\alpha' = \alpha' D
\]
\[
[D(x), \alpha^k(y)] + \varepsilon(D, x)[\alpha^k(x), D(y)] = D'([x, y]),
\]
(1.1)
for all \( x \in \text{hg}(L), y \in L \). Let \( \text{QDer}_{\alpha^k}(L) \) be the sets of homogeneous \( \alpha^k \)-quasiderivations.

**Definition 2.6** [3] Let \( (L, [\cdot, \cdot], \alpha) \) be a multiplicative Hom-Lie color algebra. If \( Z(L) := \bigoplus_{\theta \in \Gamma} Z_\theta(L), \) with \( Z_\theta(L) = \{ x \in L_\theta \mid [x, y] = 0, \forall x \in \text{hg}(L), y \in L \} \), then \( Z(L) \) is called the center of \( L \).
3 Main Results

Lemma 3.1 Let \((L, [\cdot, \cdot], \alpha)\) be a Hom-Lie color algebra over \(K\) and \(t\) an indeterminate. We define \(L_0 := L_0[tF[t]/(t^2)] = \{\Sigma(x_\alpha \otimes t + y_\beta \otimes t^2) | x_\alpha, y_\beta \in L_t\}, \) \(\bar{\alpha}(L_0) := \{\Sigma(\alpha(x_\alpha) \otimes t + \alpha(y_\beta) \otimes t^2) : x_\alpha, y_\beta \in L_t\}\), and let \(\tilde{L} = L_0 \oplus \bar{L}_1\).

Then \(\tilde{L}\) is a Hom-Lie color algebra with the operation \([x_\lambda \otimes t^i, x_\theta \otimes t^j] = [x_\lambda, x_\theta] \otimes t^{i+j}, \) for all \(x_\lambda, x_\theta \in \text{hg}(L), \) \(i, j \in \{1, 2\}\).

Proof. For all \(x_\lambda, x_\theta, x_\mu \in \text{hg}(L)\) and \(i, j, k \in \{1, 2\}\), we have

\[
[x_\lambda \otimes t^i, x_\theta \otimes t^j] = [x_\lambda, x_\theta] \otimes t^{i+j} = -\varepsilon(\lambda, \theta)[x_\theta \otimes t^j, x_\lambda \otimes t^i],
\]

\[
[\bar{\alpha}(x_\lambda \otimes t^i), [x_\theta \otimes t^j, x_\mu \otimes t^k]] = [\alpha(x_\lambda), [x_\theta, x_\mu]] \otimes t^{i+j+k} = (\varepsilon(\lambda, \theta)[\alpha(x_\theta), [x_\lambda, x_\mu]] \otimes t^{i+j+k} = \bar{\alpha}(x_\lambda \otimes t^i, x_\theta \otimes t^j, x_\mu \otimes t^k)].
\]

Hence \(\tilde{L}\) is a Hom-Lie color algebra. \(\square\)

For notational convenience, we write \(xt(xt^2)\) in place of \(x \otimes t(x \otimes t^2)\). If \(U\) is a \(\Gamma\)-graded subspace of \(L\) such that \(L = U \oplus [L, L]\), then \(\tilde{L} = Lt + Lt^2 = L + [L, L]t^2 + Ut^2,\) \(\) Now we define a map \(\varphi : \text{QDer}(L) \to \text{End}(\tilde{L})\) satisfying \(\varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2,\) where \(D, D'\) satisfy (1.1), \(a \in \text{hg}(L), b \in \text{hg}([L, L]), u \in \text{hg}(U)\) and \(d(a) = d(b) = d(u)\).

Lemma 3.2

\(1\) \(d(\varphi) = 0.\)

\(2\) \(\varphi\) is injective and \(\varphi(D)\) does not depend on the choice of \(D'.\)

\(3\) \(\varphi(\text{QDer}(L)) \subseteq \text{Der}(\tilde{L}).\)

Proof. It is clear.

\(2\) If \(\varphi(D_\lambda) = \varphi(D_\theta),\) then for all \(a \in \text{hg}(L), b \in \text{hg}([L, L])\) and \(u \in \text{hg}(U),\) we have

\[D_\lambda(a)t + D'_\lambda(b)t^2 = D_\theta(a)t + D'_\theta(b)t^2,\]

so \(D_\lambda(a) = D_\theta(a).\) Hence \(D_\lambda = D_\theta,\) and \(\varphi\) is injective.

Suppose that there exists \(D''\) such that

\[\varphi(D)(at + bt^2 + ut^2) = D(a)t + D''(b)t^2,\]

and

\([D(x, \alpha^k(y)) + \varepsilon(D, x)[\alpha^k(x), D(y)] = D''([x, y]),\]

\(\)
then we have

\[ D'(\{x, y\}) = D''(\{x, y\}), \]

thus \( D'(b) = D''(b) \). Hence

\[ \varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2 = D(a)t + D''(b)t^2, \]

which implies \( \varphi(D) \) is determined by \( D \).

(3) We have \([x, t^i, x_0t^j] = [x, x_0]t^{i+j} = 0 \), for all \( i + j \geq 3 \). Thus, to show \( \varphi(D) \in \text{Der}(\bar{L}) \), we need only to check the validness of the following equation

\[ \varphi(D)([xt, yt]) = [\varphi(D)(xt), \alpha^k(yt)] + \varepsilon(D, x)[\alpha^k(xt), \varphi(D)(yt)]. \]

For all \( x, y \in \text{hg}(L) \), we have

\[ \varphi(D)([xt, yt]) = \varphi(D)([x, y]t^2) = D'(\{x, y\})t^2 \]
\[ = [\varphi(D)(xt), \alpha^k(yt)] + \varepsilon(D, x)[\alpha^k(xt), \varphi(D)(yt)]. \]

Therefore, for all \( D \in \text{QDer}(L) \), we have \( \varphi(D) \in \text{Der}(\bar{L}) \)

\[ \square \]

Lemma 3.3 Let \((L, [\cdot, \cdot], \alpha)\) be a multiplicative Hom-Lie color algebra and \( \alpha \) a surjection. \( Z(L) = \{0\} \) and \( \bar{L}, \varphi \) are as defined above. Then \( \text{Der}(\bar{L}) = \varphi(\text{QDer}(L)) \oplus \text{ZDer}(\bar{L}) \). Proof. Since \( Z(L) = \{0\} \), we have \( Z(\bar{L}) = Lt^2 \). For all \( g \in \text{Der}(\bar{L}) \), we have \( g(Z(\bar{L})) \subseteq Z(\bar{L}) \), hence \( g(Ut^2) \subseteq g(Z(\bar{L})) \subseteq Z(\bar{L}) = Lt^2 \). Now we define a map \( f : Lt + [L, L]t^2 + Ut^2 \to Lt^2 \) by

\[ f(x) = \begin{cases} 
  g(x) \cap Lt^2, & x \in Lt; \\
  g(x), & x \in Ut^2; \\
  0, & x \in [L, L]t^2.
\end{cases} \]

Proof. It is clear that \( f \) is linear. Note that

\[ f([\bar{L}, \bar{L}]) = f([L, L]t^2) = 0, \quad [f(\bar{L}), \alpha^kL] \subseteq [Lt^2, \alpha^k(L)t + \alpha^k(L)t^2] = 0, \]

hence \( f \in \text{ZDer}(\bar{L}) \). Since

\[ (g-f)(Lt) = g(Lt) - g(Lt) \cap Lt^2 = g(Lt) - Lt^2 \subseteq Lt, \quad (g-f)(Ut^2) = 0, \]

and

\[ (g-f)([L, L]t^2) = g([\bar{L}, \bar{L}]) \subseteq [\bar{L}, \bar{L}] = [L, L]t^2, \]

there exist \( D, D' \in \text{End}(L) \) such that for all \( a \in L, b \in [L, L] \),

\[ (g-f)(at) = D(a)t, \quad (g-f)(bt^2) = D'(b)t^2. \]
Since \((g - f) \in \text{Der}(\bar{L})\) and by the definition of \(\text{Der}(\bar{L})\), we have
\[
[(g - f)(a_1t), \alpha^k(a_2t)] + \varepsilon(g - f, a_1)[\alpha^k(a_1t), (g - f)(a_2t)] = (g - f)([a_1t, a_2t]),
\]
for all \(a_1, a_2 \in L\). Hence
\[
[D(a_1), \alpha^k(a_2)] + \varepsilon(D, a_1)[\alpha^k(a_1), D(a_2)] = D'([a_1, a_2]).
\]
Thus \(D \in \text{QDer}(L)\). Therefore,
\[
g - f = \varphi(D) \in \varphi(\text{QDer}(L)) \Rightarrow \text{Der}(\bar{L}) \subseteq \varphi(\text{QDer}(L)) + \text{ZDer}(\bar{L}).
\]
By Lemma 3.2 (3) we have
\[
\text{Der}(\bar{L}) = \varphi(\text{QDer}(L)) + \text{ZDer}(\bar{L}).
\]
For all \(f \in \varphi(\text{QDer}(L)) \cap \text{ZDer}(\bar{L})\), there exists an element \(D \in \text{QDer}(L)\) such that \(f = \varphi(D)\). Then
\[
f(at + bt^2 + ut^2) = \varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2,
\]
for all \(a \in L, b \in [L, L]\).
On the other hand, \(D(a) = 0\), for all \(a \in L\) and so \(D = 0\). Hence \(f = 0\).
Therefore \(\text{Der}(\bar{L}) = \varphi(\text{QDer}(L)) \oplus \text{ZDer}(\bar{L})\) as desired. \(\square\)

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**References**


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