The $p$-Analogues of Some Inequalities for the Digamma Function

Kwara Nantomah

Department of Mathematics, University for Development Studies, Navrongo Campus, P. O. Box 24, Navrongo, UE/R, Ghana.
mykwarasoft@yahoo.com, knantomah@uds.edu.gh

Edward Prempeh

Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana.
eprempeh.cos@knust.edu.gh

Abstract

In this paper, we present the $p$-analogues of some inequalities concerning the digamma function.

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1 Introduction and Preliminaries

We begin by recalling some basic definitions involving the Gamma function.

The classical Euler’s Gamma function, $\Gamma(t)$ is commonly defined by

$$\Gamma(t) = \int_0^{\infty} e^{-x}x^{t-1} \, dx, \quad t > 0.$$ 

The digamma function, $\psi(t)$ is defined as the logarithmic derivative of the Gamma function, that is,

$$\psi(t) = \frac{d}{dt} \ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \quad t > 0.$$
The $p$-analogue of the Gamma function, $\Gamma_p(t)$ is defined by (see [1],[2])
\[
\Gamma_p(t) = \frac{p!p^t}{t(t+1)\ldots(t+p)} = \frac{p^t}{t(1+\frac{1}{p})\ldots(1+\frac{1}{p})}, \quad p \in \mathbb{N}, \quad t > 0.
\]
Similarly, the $p$-digamma function, $\psi_p(t)$ is defined as,
\[
\psi_p(t) = \frac{d}{dt} \ln(\Gamma_p(t)) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0.
\]
The functions $\psi(t)$ and $\psi_p(t)$ as defined above have the following series representations.
\[
\psi(t) = -\gamma + (t - 1) \sum_{n=0}^{\infty} \frac{1}{(1+n)(n+t)}, \quad t > 0.
\]
\[
\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t}, \quad p \in \mathbb{N}, \quad t > 0.
\]
where $\gamma$ is the Euler-Mascheroni’s constant. For some properties of these functions, see [4], [2] and the references therein.

By taking the $m$-th derivative of the functions $\psi(t)$ and $\psi_p(t)$, it can be shown that the following statements are valid for $m \in \mathbb{N}$.
\[
\psi^{(m)}(t) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+t)^{m+1}}, \quad t > 0.
\]
\[
\psi_p^{(m)}(t) = (-1)^{m-1} m! \sum_{n=0}^{p} \frac{1}{(n+t)^{m+1}}, \quad p \in \mathbb{N}, \quad t > 0.
\]
In a recent paper [3], Sulaiman presented the following results.
\[
\psi(s+t) \geq \psi(s) + \psi(t) \quad (1)
\]
where $t > 0$ and $0 < s < 1$.
\[
\psi^{(m)}(s+t) \leq \psi^{(m)}(s) + \psi^{(m)}(t) \quad (2)
\]
where $m$ is a positive odd integer and $s, t > 0$.
\[
\psi^{(m)}(s+t) \geq \psi^{(m)}(s) + \psi^{(m)}(t) \quad (3)
\]
where $m$ is a positive even integer and $s, t > 0$.

The objective of this paper is to establish that the inequalities (1), (2) and (3) still hold true for the function $\psi_p(t)$. 

2 Main Results

We now present the results of this paper.

**Theorem 2.1.** Let $t > 0$, $0 < s \leq 1$ and $p \in \mathbb{N}$. Then the following inequality holds true.

$$
\psi_p(s + t) \geq \psi_p(s) + \psi_p(t). \quad (4)
$$

**Proof.** Let $f(t) = \psi_p(s + t) - \psi_p(s) - \psi_p(t)$. Then fixing $s$ we have,

$$
f'(t) = \psi'_p(s + t) - \psi'_p(t) = \sum_{n=0}^{p} \frac{1}{(n + s + t)^2} - \sum_{n=0}^{p} \frac{1}{(n + t)^2}
= \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^2} - \frac{1}{(n + t)^2} \right] \leq 0.
$$

That implies $f$ is non-increasing. Further,

$$
\lim_{t \to \infty} f(t) = \lim_{t \to \infty} \left[ \psi_p(s + t) - \psi_p(s) - \psi_p(t) \right]
= \lim_{t \to \infty} \left[ - \ln p + \sum_{n=0}^{p} \frac{1}{(n + s + t)} + \sum_{n=0}^{p} \frac{1}{(n + t)} - \sum_{n=0}^{p} \frac{1}{(n + s + t)} \right]
= - \ln p + \lim_{t \to \infty} \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)} + \frac{1}{(n + t)} - \frac{1}{(n + s + t)} \right]
= - \ln p + \sum_{n=0}^{p} \frac{1}{(n + s)} \geq 0.
$$

Therefore $f(t) \geq 0$ yielding the result.

**Theorem 2.2.** Let $s, t > 0$ and $p \in \mathbb{N}$. Suppose that $m$ is a positive odd integer, then the following inequality holds true.

$$
\psi_p^{(m)}(s + t) \leq \psi_p^{(m)}(s) + \psi_p^{(m)}(t). \quad (5)
$$

**Proof.** Let $g(t) = \psi_p^{(m)}(s + t) - \psi_p^{(m)}(s) - \psi_p^{(m)}(t)$. Then fixing $s$ we have,

$$
g'(t) = \psi_p^{(m+1)}(s + t) - \psi_p^{(m+1)}(t)
= (-1)^m(m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+2}} - \frac{1}{(n + t)^{m+2}} \right]
= -(m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+2}} - \frac{1}{(n + t)^{m+2}} \right], \text{ (since } m \text{ is odd)}
= (m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + t)^{m+2}} - \frac{1}{(n + s + t)^{m+2}} \right] \geq 0.
$$
That implies \( g \) is non-decreasing. Further,

\[
\lim_{t \to \infty} g(t) = \lim_{t \to \infty} \left[ \psi_p^{(m)}(s + t) - \psi_p^{(m)}(s) - \psi_p^{(m)}(t) \right]
\]

\[
= (-1)^{m-1}m! \lim_{t \to \infty} \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+1}} - \frac{1}{(n + s)^{m+1}} - \frac{1}{(n + t)^{m+1}} \right]
\]

\[
= m! \lim_{t \to \infty} \sum_{n=0}^{p} \frac{1}{(n + s + t)^{m+1}} - \frac{1}{(n + s)^{m+1}} - \frac{1}{(n + t)^{m+1}}, \text{ (for odd } m \text{)}
\]

\[
= -m! \sum_{n=0}^{p} \frac{1}{(n + s)^{m+1}} \leq 0.
\]

Therefore \( g(t) \leq 0 \) yielding the result.

**Theorem 2.3.** Let \( s, t > 0 \) and \( p \in N \). Suppose that \( m \) is a positive even integer, then the following inequality holds true.

\[
\psi_p^{(m)}(s + t) \geq \psi_p^{(m)}(s) + \psi_p^{(m)}(t).
\] (6)

**Proof.** Let \( h(t) = \psi_p^{(m)}(s + t) - \psi_p^{(m)}(s) - \psi_p^{(m)}(t) \). Then by fixing \( s \) we have,

\[
h'(t) = \psi_p^{(m+1)}(s + t) - \psi_p^{(m+1)}(t)
\]

\[
= (-1)^{m}(m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+2}} - \frac{1}{(n + t)^{m+2}} \right]
\]

\[
= (m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+2}} - \frac{1}{(n + t)^{m+2}} \right], \text{ (since } m \text{ is even)}
\]

\[
= (m + 1)! \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+2}} - \frac{1}{(n + t)^{m+2}} \right] \leq 0.
\]

That implies \( h \) is non-increasing. Further,

\[
\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \left[ \psi_p^{(m)}(s + t) - \psi_p^{(m)}(s) - \psi_p^{(m)}(t) \right]
\]

\[
= (-1)^{m-1}m! \lim_{t \to \infty} \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+1}} - \frac{1}{(n + s)^{m+1}} - \frac{1}{(n + t)^{m+1}} \right]
\]

\[
= -m! \lim_{t \to \infty} \sum_{n=0}^{p} \left[ \frac{1}{(n + s + t)^{m+1}} - \frac{1}{(n + s)^{m+1}} - \frac{1}{(n + t)^{m+1}} \right], \text{ (for even } m \text{)}
\]

\[
= m! \sum_{n=0}^{p} \frac{1}{(n + s)^{m+1}} \geq 0.
\]

Therefore \( h(t) \geq 0 \) yielding the result.
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References


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