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The Classification of 4-dimensional Leibniz Superalgebras

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Abstract

In this paper ,we study the classification of 4-dimensional Leibniz superalgebras by using the method of undetermined coefficients , the definition of Leibniz superalgebras and Jordan's normal form. Furthermore ,we give the classification of a specific case of 4-dimensional Leibniz superalgebras.

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1 Introduction

Leibniz superalgebras was first mentioned by A. Bloch as D-algebra involved in the literature[1].Then J. L. Loday formally put forward this concept.The classification of some concrete algebras is what many scholars have been trying to do. In the classification of non-Lie's Leibniz algebras, the classification of 2- dimensional Leibniz algebras, 3-dimensional Leibniz algebras and 4-dimensional nilpotent Leibniz algebras have been finished. On the basis of the existing classification, this paper aims to classify the 4-dimensional Leibniz superalgebras according to the definition of Leibniz superalgebras and Jordan's normal form.

The paper is organized as follows. In Section 2, we recall the definition of Leibniz algebras and Leibniz superalgebras.

In Section 3, we will give the classification of a specific case of 4-dimensional Leibniz superalgebras.

2 Preliminary Notes

First we present the definition of Leibniz algebras.

Definition 2.1 *A Leibniz superalgebras L is a Z_2-graded algebra $L = L_{\bar{0}} + L_{\bar{1}}$ over an arbitrary base field K, endowed with a product $[-,-]$, which satisfies the following conditions:*

(1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$,*for any* α,$\beta \in Z_2$

(2) $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$,*for any* $x \in L$,$y \in L_\alpha$,$z \in L_\beta$,$\alpha, \beta \in Z_2$

Definition 2.2 *A Z_2-graded vector space $L = L_{\bar{0}} + L_{\bar{1}}$ is called a Lie superalgebras if it is equipped with a product $[-,-]$ which satisfies the following conditions:*

(1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$,*for any* α,$\beta \in Z_2$

(2) $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$,*for any* $x \in L$,$y \in L_\alpha$,$z \in L_\beta$,$\alpha, \beta \in Z_2$

(3) $[x,y] = -(-1)^{\alpha\beta}[y,x]$,*for any* $x \in L_\alpha$,$y \in L_\beta$

3 Main Results

Since $L = L_{\bar{0}} + L_{\bar{1}}$,If $dimL = 4$, there are five possibilities:

(1) $dimL_{\bar{0}} = 0$,$dimL_{\bar{1}} = 4$

(2) $dimL_{\bar{0}} = 3$,$dimL_{\bar{1}} = 1$

(3) $dimL_{\bar{0}} = 1$,$dimL_{\bar{1}} = 3$

(4) $dimL_{\bar{0}} = 2$,$dimL_{\bar{1}} = 2$

(5) $dimL_{\bar{0}} = 4$,$dimL_{\bar{1}} = 0$

As to the first two situations, we can use the definition of Leibniz superalgebra and the permutation and combination of bases to conduct the classification. As to the last three situations, we can take advantage of Jordan's normal form to conduct the classification. Here, we take $dimL_{\bar{0}} = 1$,$dimL_{\bar{1}} = 3$ as an example.

Lemma 3.1 *Assume L is a non-Lie superalgebra's 4-dimensional Leibniz superalgebra, e_0,e_1,e_2,e_3 are bases of L.e_0 is the basis of $L_{\bar{0}}$.e_1,e_2,e_3 are bases of $L_{\bar{1}}$.There is a transform between them denoted by $ade_0(e_1,e_2,e_3) = (e_1,e_2,e_3)A$.Through the basis transformation, A is finally similar to the following 11 types:*

$$(1)\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\ (2)\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\ (3)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\ (4)\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$(5)\begin{pmatrix}0&1&0\\0&0&0\\0&0&1\end{pmatrix}\ (6)\begin{pmatrix}1&1&0\\0&1&0\\0&0&0\end{pmatrix}\ (7)\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\ (8)\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}$$

$$(9)\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\ (10)\begin{pmatrix}1&0&0\\0&a&0\\0&0&a\end{pmatrix}(a\neq 0)\ (11)\begin{pmatrix}1&0&0\\0&a&0\\0&0&b\end{pmatrix}(a\neq b\neq 0)$$

Lemma 3.2 *For any $x \in L_{\bar{1}}$, assume $[e_1,x] = f(x)e_0$ and $[x,e_1] = g(x)e_0$,then $fg \in L_{\bar{1}}^*$ and $dim(Kerf) + dim(lmf) = 3$,$dim(Kerg) + dim(lmg) = 3$.*

Lemma 3.3 *Assume L is a non-Lie superalgebra's 4-dimensional Leibniz superalgebra, e_0,e_1,e_2,e_3 are bases of L.e_0 is the basis of $L_{\bar{0}}$.e_1,e_2,e_3 are bases of $L_{\bar{1}}$.There is a transform between them denoted by $(e_1,e_2,e_3)Re_0 = (e_1,e_2,e_3)B$.Through the basis transformation, B is finally similar to the 11 types in Lemma3.1*

Theorem 3.4 *Assume L is a 4-dimensional Leibniz superalgebra,and $dimL_{\bar{0}} = 1$,$dimL_{\bar{1}} = 3$,e_0,e_1,e_2,e_3 are bases of L.e_0 is the basis of $L_{\bar{0}}$. e_1,e_2,e_3 are bases of $L_{\bar{1}}$,there are 29 kinds of 4-dimensional Leibniz superalgebras:*

(1) $[e_1,e_2] = -[e_2,e_1] = e_0, [e_1,e_3] = -[e_3,e_1] = \lambda_1 e_0, [e_1,e_2] = -[e_2,e_1] = \lambda_2 e_0$, *other products are zero.*

(2) $[e_1,e_1] = e_0, [e_3,e_1] = \lambda_1 e_0, [e_3,e_3] = \lambda_2 e_0, [e_2,e_1] = [e_2,e_3] = [e_3,e_1] = [e_3,e_2] = e_0$, *other products are zero.*

(3) $[e_1,e_0] = e_1$, *other products are zero.*

(4) $[e_0,e_1] = e_1, [e_1,e_0] = -e_1, [e_2,e_0] = be_2 + ce_3, [e_3,e_0] = de_2 + ee_3$,*other products are zero.*

(5) $[e_0,e_1] = e_1, [e_0,e_2] = ae_2, [e_0,e_3] = be_3$, *other products are zero.*

(6) $[e_1,e_0] = -e_1, [e_2,e_0] = -ae_2, [e_3,e_0] = -be_3$, *other products are zero.*

(7) $[e_0,e_2] = e_1, [e_2,e_0] = k_1 e_1 + k_3 e_3$, *other products are zero.*

(8) $[e_0, e_2] = e_1, [e_2, e_0] = k_3e_3,$ *other products are zero.*

(9) $[e_0, e_2] = e_1, [e_2, e_0] = k_1e_1,$ *other products are zero.*

(10) $[e_0, e_2] = e_1, [e_2, e_2] = e_0, [e_2, e_0] = 2e_1,$ *other products are zero.*

(11) $[e_0, e_1] = -[e_1, e_0] = e_1, [e_0, e_2] = -[e_2, e_0] = e_1 + e_2, [e_3, e_0] = k_3e_3(k_3 \neq 0),$*other products are zero.*

(12) $[e_0, e_2] = e_1, [e_2, e_2] = ae_1(a \neq 0), [e_2, e_0] = 2e_1 + k_3e_3(k_3 \neq 0), [e_3, e_0] = l_1e_1 + l_3e_3,$*other products are zero.*

(13) $[e_0, e_1] = e_1, [e_2, e_2] = ae_0(a \neq 0), [e_2, e_0] = 2e_1, [e_3, e_0] = l_1e_1 + l_3e_3,$*other products are zero.*

(14) $[e_0, e_1] = e_1, [e_2, e_0] = k_1e_1 + k_3e_3,$ *other products are zero.*

(15) $[e_0, e_1] = e_1, [e_1, e_0] = -e_1, [e_2, e_0] = be_2 + ce_3, [e_3, e_0] = de_2 + ee_3$ *(b, c, d, e are not all zero) , other products are zero.*

(16) $[e_0, e_1] = e_1, [e_0, e_2] = ae_2, [e_0, e_3] = ae_3, [e_1, e_0] = -e_1, [e_2, e_0] = k_1e_2 + k_2e_3, [e_3, e_0] = k_3e_2 + k_4e_3,$*($k_1 \neq k_4, k_2, k_3$ are not all zero) , other products are zero.*

(17) $[e_0, e_1] = e_1, [e_0, e_2] = e_1 + e_2, [e_0, e_3] = e_2 + e_3, [e_1, e_0] = -e_1, [e_2, e_0] = k_1e_1 - e_2, [e_3, e_0] = k_2e_1 + k_1e_2 - e_3,$ $(k_1 \neq -1, k_2 \neq 0),$ *other products are zero.*

(18) $[e_0, e_2] = e_1, [e_0, e_3] = e_3, [e_3, e_0] = -e_3, [e_2, e_0] = xe_1,$ *other products are zero.*

(19) $[e_0, e_2] = e_1, [e_0, e_3] = e_3, [e_3, e_0] = -e_3, [e_2, e_0] = 2e_1, [e_2, e_2] = be_0(b \neq 0),$*other products are zero.*

(20) $[e_0, e_2] = e_1, [e_0, e_3] = e_2, [e_2, e_0] = -e_1,$ *other products are zero.*

(21) $[e_1, e_0] = e_1, [e_2, e_0] = e_2,$ *other products are zero.*

(22) $[e_1, e_0] = e_1, [e_2, e_0] = ae_2, [e_3, e_0] = ae_3(a \neq 0),$ *other products are zero.*

(23) $[e_1, e_0] = e_1, [e_2, e_0] = ae_2, [e_3, e_0] = be_3(a \neq b \neq 0),$ *other products are zero.*

(24) $[e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1,$ *other products are zero.*

(25) $[e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1, [e_3, e_0] = e_3,$ *other products are zero.*

(26) $[e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1, [e_3, e_0] = e_2 + e_3,$ *other products are zero.*

(27) $[e_2, e_0] = e_1$, *other products are zero.*

(28) $[e_2, e_0] = e_1$, *other products are zero.*

(29) $[e_2, e_0] = e_1, [e_3, e_0] = e_2$, *other products are zero.*

Proof.(1) $Let A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

In this case, bases could be chosen arbitrarily. So B can be finally similar to the 11 types Jordan's normal form mentioned in Lemma3.3 by choosing right bases.

Assume that $[e_1, e_1] = a_1e_0, [e_1, e_2] = b_1e_0, [e_1, e_3] = c_1e_0$, $[e_2, e_1] = a_2e_0$, $[e_2, e_2] = b_2e_0, [e_2, e_3] = c_2e_0$, $[e_3, e_1] = a_3e_0, [e_3, e_2] = b_3e_0, [e_3, e_3] = c_3e_0$.

(I) $When B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,In fact, it is not difficult to see that the following inequality holds, $a_2 \neq b_1, a_3 \neq c_1, b_3 \neq c_2$

(i) If $\forall e_i \in L_{\bar{0}}$, $[e_i, e_i] = 0$. We can obtain $[e_i, e_j] = -[e_j, e_i]$. In this case, it is clear that there is only one kind of non- Lie superalgebra's 4-dimensional Leibniz superalgebra satisfy conditions. That is $[e_1, e_2] = -[e_2, e_1] = e_0, [e_1, e_3] = -[e_3, e_1] = \lambda_1 e_0, [e_1, e_2] = -[e_2, e_1] = \lambda_2 e_0$, other products are zero.

(ii) If $e_i \in L_{\bar{1}}, [e_i, e_i] \neq 0$. We can assume$[e_1, e_1] = e_0$.It can be noticed that $\forall x \in L_{\bar{1}}$, $[e_i, x] = f(x)e_0$,$dim(Kerf)$+$dim(lmf) = 3$, then $dim(Kerf) =$ 2.So $e_2, e_3 \in L_{\bar{1}}$, $[e_1, e_2] = 0$,$[e_1, e_3] = 0$, and e_1, e_2, e_3 are the bases of $L_{\bar{1}}$,then it is clear that there is only one kind of non- Lie superalgebra's 4-dimensional Leibniz superalgebra satisfy conditions. $[e_1, e_1] = e_0, [e_3, e_1] = \lambda_1 e_0, [e_3, e_3] = \lambda_2 e_0, [e_2, e_1] = [e_2, e_3] = [e_3, e_1] = [e_3, e_2] = e_0$, other products are zero.

$(II) When B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

In this case, $[e_1, e_0] = e_1$,$[e_2, e_0] = 0$,$[e_3, e_0] = 0$.

$[e_1, [e_1, e_1]] = [[e_1, e_1], e_1] + [[e_1, e_1], e_1] = 2[[e_1, e_1], e_1] \Rightarrow a_1 = 0, [e_1, e_1] = 0$

$[e_1, [e_2, e_2]] = 2[[e_1, e_2], e_2] = 2b_1[e_0, e_2] \Rightarrow b_2 = 0, [e_2, e_2] = 0$

$[e_1, [e_3, e_3]] = 2[[e_1, e_3], e_3] = 2c_1[e_0, e_3] = 0 \Rightarrow c_3 = 0, [e_3, e_3] = 0$

$[e_1, [e_0, e_2]] = [[e_1, e_0], e_2] - [[e_1, e_2], e_0] \Rightarrow [e_1, e_2] = 0$

$[e_1, [e_0, e_3]] = [[e_1, e_0], e_3] - [[e_1, e_3], e_0] \Rightarrow [e_1, e_3] = 0$

$[e_2, [e_1, e_0]] = [[e_2, e_1], e_0] - [[e_2, e_0], e_1] \Rightarrow [e_2, e_1] = 0$

$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [[e_1, e_3], e_2] = 0 \Rightarrow c_2 = 0, [e_2, e_3] = 0$

$[e_1, [e_3, e_1]] = [[e_1, e_3], e_1] + [[e_1, e_1], e_3] = 0 \Rightarrow a_3 = 0, [e_3, e_1] = 0$

$[e_1, [e_3, e_2]] = [[e_1, e_3], e_2] + [[e_1, e_2], e_3] = 0 \Rightarrow b_3 = 0, [e_3, e_2] = 0$

In summary,there is only one kind of non- Lie superalgebra's 4-dimensional Leibniz superalgebra,$[e_1, e_0] = e_1$, other products are zero.

(2) $LetA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

In this case, $[e_0, e_1] = e_1, [e_0, e_2] = 0, [e_0, e_3] = 0$.

Similarly,there is only one kind of non-Lie superalgebra's 4-dimensional Leibniz superalgebra,$[e_0, e_1] = e_1, [e_1, e_0] = -e_1, [e_2, e_0] = be_2 + ce_3, [e_3, e_0] = de_2 + ee_3$,other products are zero.While b, c, d, e are not all zero,L is non-Lie superalgebra's Leibniz superalgebras.

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