The Classification of 4-dimensional Leibniz Superalgebras

Qi Nie

School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, CHINA

Zhen Li

School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, CHINA

Abstract

In this paper, we study the classification of 4-dimensional Leibniz superalgebras by using the method of undetermined coefficients, the definition of Leibniz superalgebras and Jordan’s normal form. Furthermore, we give the classification of a specific case of 4-dimensional Leibniz superalgebras.

Mathematics Subject Classification (2010): 16S70, 17A42, 17B10, 17B56, 17B70

Keywords: Leibniz superalgebras, Jordan’s normal form, 4-dimensional, classification

1 Introduction

Leibniz superalgebras was first mentioned by A. Bloch as D-algebra involved in the literature[1]. Then J. L. Loday formally put forward this concept. The classification of some concrete algebras is what many scholars have been trying to do. In the classification of non-Lie’s Leibniz algebras, the classification of 2-dimensional Leibniz algebras, 3-dimensional Leibniz algebras and 4-dimensional nilpotent Leibniz algebras have been finished. On the basis of the existing classification, this paper aims to classify the 4-dimensional Leibniz superalgebras according to the definition of Leibniz superalgebras and Jordan’s normal form.

The paper is organized as follows. In Section 2, we recall the definition of Leibniz algebras and Leibniz superalgebras.
In Section 3, we will give the classification of a specific case of 4-dimensional Leibniz superalgebras.

2 Preliminary Notes

First we present the definition of Leibniz algebras.

Definition 2.1 A Leibniz superalgebra $L$ is a $\mathbb{Z}_2$-graded algebra $L = L_0 + L_1$ over an arbitrary base field $K$, endowed with a product $[\cdot, \cdot]$, which satisfies the following conditions:

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for any $\alpha, \beta \in \mathbb{Z}_2$
2. $[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta} [[x, z], y]$, for any $x \in L, y \in L_\alpha, z \in L_\beta, \alpha, \beta \in \mathbb{Z}_2$

Definition 2.2 A $\mathbb{Z}_2$-graded vector space $L = L_0 + L_1$ is called a Lie superalgebra if it is equipped with a product $[\cdot, \cdot]$ which satisfies the following conditions:

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for any $\alpha, \beta \in \mathbb{Z}_2$
2. $[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta} [[x, z], y]$, for any $x \in L, y \in L_\alpha, z \in L_\beta, \alpha, \beta \in \mathbb{Z}_2$
3. $[x, y] = -(-1)^{\alpha\beta} [y, x]$, for any $x \in L_\alpha, y \in L_\beta$

3 Main Results

Since $L = L_0 + L_1$, if $\dim L = 4$, there are five possibilities:

1. $\dim L_0 = 0, \dim L_1 = 4$
2. $\dim L_0 = 3, \dim L_1 = 1$
3. $\dim L_0 = 1, \dim L_1 = 3$
4. $\dim L_0 = 2, \dim L_1 = 2$
5. $\dim L_0 = 4, \dim L_1 = 0$

As to the first two situations, we can use the definition of Leibniz superalgebra and the permutation and combination of bases to conduct the classification. As to the last three situations, we can take advantage of Jordan’s normal form to conduct the classification. Here, we take $\dim L_0 = 1, \dim L_1 = 3$ as an example.
The Classification of 4-dimensional Leibniz Superalgebras

Lemma 3.1 Assume $L$ is a non-Lie superalgebra’s 4-dimensional Leibniz superalgebra, $e_0,e_1,e_2,e_3$ are bases of $L$. $e_0$ is the basis of $L_0,e_1,e_2,e_3$ are bases of $L_1$. There is a transform between them denoted by $ad e_0(e_1,e_2,e_3) = (e_1,e_2,e_3)A$. Through the basis transformation, $A$ is finally similar to the following 11 types:

\[
\begin{align*}
(1) & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(2) & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(3) & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(4) & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(5) & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(6) & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(7) & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
(8) & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\
(9) & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
(10) & \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ a \neq 0 \\ a \neq b \neq 0 \end{pmatrix}, \\
(11) & \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}.
\end{align*}
\]

Lemma 3.2 For any $x \in L_1$, assume $[e_1,x] = f(x)e_0$ and $[x,e_1] = g(x)e_0$, then $fg \in L_1$ and $\dim(Ker f) + \dim(Im f) = 3, \dim(Ker g) + \dim(Im g) = 3$.

Lemma 3.3 Assume $L$ is a non-Lie superalgebra’s 4-dimensional Leibniz superalgebra, $e_0,e_1,e_2,e_3$ are bases of $L$. $e_0$ is the basis of $L_0,e_1,e_2,e_3$ are bases of $L_1$. There is a transform between them denoted by $(e_1,e_2,e_3)Ae_0 = (e_1,e_2,e_3)B$. Through the basis transformation, $B$ is finally similar to the 11 types in Lemma 3.1.

Theorem 3.4 Assume $L$ is a 4-dimensional Leibniz superalgebra, and $\dim L_0 = 1, \dim L_1 = 3, e_0,e_1,e_2,e_3$ are bases of $L$. $e_0$ is the basis of $L_0$. $e_1,e_2,e_3$ are bases of $L_1$, there are 29 kinds of 4-dimensional Leibniz superalgebras:

\[
\begin{align*}
(1) & \ [e_1,e_2] = -[e_2,e_1] = e_0, [e_1,e_3] = -[e_3,e_1] = \lambda_1 e_0, [e_1,e_2] = -[e_2,e_1] = \lambda_2 e_0, \text{other products are zero}. \\
(2) & \ [e_1,e_1] = e_0, [e_3,e_1] = \lambda_1 e_0, [e_3,e_3] = \lambda_2 e_0, [e_2,e_1] = [e_2,e_3] = [e_3,e_1] = [e_3,e_2] = e_0, \text{other products are zero}. \\
(3) & \ [e_1,e_0] = e_1, \text{other products are zero}. \\
(4) & \ [e_0,e_1] = e_1, [e_1,e_0] = -e_1, [e_2,e_0] = be_2 + ce_3, [e_3,e_0] = de_2 + ee_3, \text{other products are zero}. \\
(5) & \ [e_0,e_1] = e_1, [e_0,e_2] = ae_2, [e_0,e_3] = be_3, \text{other products are zero}. \\
(6) & \ [e_1,e_0] = -e_1, [e_2,e_0] = -ae_2, [e_3,e_0] = -be_3, \text{other products are zero}. \\
(7) & \ [e_0,e_2] = e_1, [e_2,e_0] = k_1 e_1 + k_3 e_3, \text{other products are zero}.
\end{align*}
\]
(8) \( [e_0, e_2] = e_1, [e_2, e_0] = k_3 e_3, \) other products are zero.

(9) \( [e_0, e_2] = e_1, [e_2, e_0] = k_1 e_1, \) other products are zero.

(10) \( [e_0, e_2] = e_1, [e_2, e_2] = e_0, [e_2, e_0] = 2 e_1, \) other products are zero.

(11) \( [e_0, e_1] = -[e_1, e_0] = e_1, [e_0, e_2] = -[e_2, e_0] = e_1 + e_2, [e_3, e_0] = k_3 e_3(k_3 \neq 0), \) other products are zero.

(12) \( [e_0, e_2] = e_1, [e_2, e_2] = a e_1(a \neq 0), [e_2, e_0] = 2 e_1 + k_3 e_3(k_3 \neq 0), [e_3, e_0] = l_1 e_1 + l_3 e_3, \) other products are zero.

(13) \( [e_0, e_1] = e_1, [e_2, e_2] = a e_0(a \neq 0), [e_2, e_0] = 2 e_1, [e_3, e_0] = l_1 e_1 + l_3 e_3, \) other products are zero.

(14) \( [e_0, e_1] = e_1, [e_2, e_0] = k_1 e_1 + k_3 e_3, \) other products are zero.

(15) \( [e_0, e_1] = e_1, [e_1, e_0] = -e_1, [e_2, e_0] = be_2 + ce_3, [e_3, e_0] = de_2 + ee_3 (b, c, d, e are not all zero), \) other products are zero.

(16) \( [e_0, e_1] = e_1, [e_0, e_2] = a e_2, [e_0, e_3] = a e_3, [e_1, e_0] = -e_1, [e_2, e_0] = k_1 e_2 + k_2 e_3, [e_3, e_0] = k_3 e_2 + k_4 e_3, (k_1 \neq k_4, k_2, k_3 are not all zero), \) other products are zero.

(17) \( [e_0, e_1] = e_1, [e_0, e_2] = e_1 + e_2, [e_0, e_3] = e_2 + e_3, [e_1, e_0] = -e_1, [e_2, e_0] = k_1 e_1 - e_2, [e_3, e_0] = k_2 e_1 + k_1 e_2 - e_3, (k_1 \neq -1, k_2 \neq 0), \) other products are zero.

(18) \( [e_0, e_2] = e_1, [e_0, e_3] = e_3, [e_3, e_0] = -e_3, [e_2, e_0] = xe_1, \) other products are zero.

(19) \( [e_0, e_2] = e_1, [e_0, e_3] = e_3, [e_3, e_0] = -e_3, [e_2, e_0] = 2 e_1, [e_2, e_2] = be_0(b \neq 0), \) other products are zero.

(20) \( [e_0, e_2] = e_1, [e_0, e_3] = e_2, [e_2, e_0] = -e_1, \) other products are zero.

(21) \( [e_1, e_0] = e_1, [e_2, e_0] = e_2, \) other products are zero.

(22) \( [e_1, e_0] = e_1, [e_2, e_0] = a e_2, [e_3, e_0] = a e_3(a \neq 0), \) other products are zero.

(23) \( [e_1, e_0] = e_1, [e_2, e_0] = a e_2, [e_3, e_0] = b e_3(a \neq b \neq 0), \) other products are zero.

(24) \( [e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1, \) other products are zero.

(25) \( [e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1, [e_3, e_0] = e_3, \) other products are zero.

(26) \( [e_1, e_0] = e_1, [e_2, e_0] = e_2 + e_1, [e_3, e_0] = e_2 + e_3, \) other products are zero.
The Classification of 4-dimensional Leibniz Superalgebras

(27) \([e_2, e_0] = e_1, \) other products are zero.

(28) \([e_2, e_0] = e_1, \) other products are zero.

(29) \([e_2, e_0] = e_1, [e_3, e_0] = e_2, \) other products are zero.

Proof. (1) Let \( A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

In this case, bases could be chosen arbitrarily. So \( B \) can be finally similar to the 11 types Jordan’s normal form mentioned in Lemma 3.3 by choosing right bases.

Assume that \([e_1, e_1] = a_1 e_0, [e_1, e_2] = b_1 e_0, [e_1, e_3] = c_1 e_0, [e_2, e_1] = a_2 e_0,\)

\([e_2, e_2] = b_2 e_0, [e_2, e_3] = c_2 e_0, [e_3, e_1] = a_3 e_0, [e_3, e_2] = b_3 e_0, [e_3, e_3] = c_3 e_0.\)

(I) When \( B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

In fact, it is not difficult to see that the following inequality holds, \( a_2 \neq b_1, a_3 \neq c_1, b_3 \neq c_2.\)

(i) If \( \forall e_i \in L_0, [e_i, e_i] = 0.\) We can obtain \([e_i, e_j] = [e_j, e_i].\) In this case, it is clear that there is only one kind of non-Lie superalgebra’s 4-dimensional Leibniz superalgebra satisfy conditions. That is \([e_1, e_2] = [e_2, e_1] = e_0, [e_1, e_3] = [e_3, e_1] = e_0, [e_2, e_1] = e_0, [e_2, e_3] = e_0, other products are zero.

(ii) If \( e_i \in L_1, [e_i, e_i] \neq 0.\) We can assume \([e_1, e_1] = e_0.\) It can be noticed that \( \forall x \in L_1, [e_i, x] = f(x) e_0.\) Then \( \dim(\ker f) + \dim(\text{Im} f) = 3,\) then \( \dim(\ker f) = 2.\) So \( e_2, e_3 \in L_1, [e_1, e_2] = 0, [e_1, e_3] = 0, \) and \( e_1, e_2, e_3 \) are the bases of \( L_1,\) then it is clear that there is only one kind of non-Lie superalgebra’s 4-dimensional Leibniz superalgebra satisfy conditions. \([e_1, e_1] = e_0, [e_3, e_1] = e_1, [e_2, e_1] = e_2, [e_3, e_2] = e_3, [e_3, e_3] = e_0, other products are zero.

(II) When \( B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

In this case, \([e_1, e_0] = e_1, \) \([e_2, e_0] = 0, [e_3, e_0] = 0.\)

\([e_1, [e_1, e_1]] = [[e_1, e_1], e_1] + [[e_1, e_1], e_1] = 2[[e_1, e_1], e_1] \Rightarrow a_1 = 0, [e_1, e_1] = 0 \)

\([e_1, [e_2, e_2]] = 2[[e_1, e_2], e_2] = 2b_1 e_0 \Rightarrow b_2 = 0, [e_2, e_2] = 0 \)

\([e_1, [e_1, e_3]] = 2[[e_1, e_3], e_3] = 2c_1 e_0 \Rightarrow c_3 = 0, [e_3, e_3] = 0 \)

\([e_1, [e_0, e_2]] = [[e_1, e_0], e_2] - [[e_1, e_2], e_0] \Rightarrow [e_1, e_2] = 0 \)

\([e_1, [e_0, e_3]] = [[e_1, e_0], e_3] - [[e_1, e_3], e_0] \Rightarrow [e_1, e_3] = 0 \)

\([e_2, [e_1, e_0]] = [[e_2, e_1], e_0] - [[e_2, e_0], e_1] \Rightarrow [e_2, e_1] = 0 \)

\([e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [[e_1, e_3], e_2] = 0 \Rightarrow c_2 = 0, [e_2, e_3] = 0 \)

\([e_1, [e_3, e_1]] = [[e_1, e_3], e_1] + [[e_1, e_1], e_3] = 0 \Rightarrow a_3 = 0, [e_3, e_1] = 0 \)

\([e_1, [e_3, e_2]] = [[e_1, e_3], e_2] + [[e_1, e_2], e_3] = 0 \Rightarrow b_3 = 0, [e_3, e_2] = 0 \)

In summary, there is only one kind of non-Lie superalgebra’s 4-dimensional Leibniz superalgebra, \([e_1, e_0] = e_1,\) other products are zero.
Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

In this case, $[e_0, e_1] = e_1$, $[e_0, e_2] = 0$, $[e_0, e_3] = 0$.

Similarly, there is only one kind of non-Lie superalgebra’s 4-dimensional Leibniz superalgebra, $[e_0, e_1] = e_1$, $[e_1, e_0] = -e_1$, $[e_2, e_0] = be_2 + ce_3$, $[e_3, e_0] = de_2 + ee_3$, other products are zero. While $b, c, d, e$ are not all zero, $L$ is non-Lie superalgebra’s Leibniz superalgebras.

**ACKNOWLEDGEMENTS.**

First of all, I would like to extend my sincere gratitude to my tutor, Mr Chen, for his instructive advice and useful suggestions on my thesis. I am deeply grateful of his help in the completion of this thesis. I am also deeply indebted to all the other partners and teachers in The Classification of 4-dimensional Leibniz Superalgebras for their direct and indirect help to me. Finally, I am indebted to my parents for their continuous support and encouragement.

**References**


Received: December 05, 2017