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Supplements to a class of logarithmically completely monotonic functions related to the $q-$gamma function

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Abstract

In this paper, we investigate necessary and sufficient conditions for logarithmic complete monotonicity of a class of functions related to $q-$gamma function . Some results of the paper generalize those due to Guo, Qi, and Srivastava [1]. As consequences of these results, supplements to the recent investigation by the author [2] are provided and the $q-$version of Kěckić-Vasić type inequality is established.

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1 Introduction

Recall that a non-negative function f defined on $(0,\infty)$ is called completely monotonic if it has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geq 0,\ n \geq 1$$

and $x > 0$ [[3], Def. 1.3]. This inequality is known to be strict unless f is a constant. By the celebrated Bernstein theorem, a function is completely monotonic if and only if it is the Laplace transform of a non-negative measure [[3], Thm. 1.4]. The above definition implies the following equivalences:

$$f \text{ is CM on}(0,\infty) \Leftrightarrow f \geq 0, -f' \text{ is completely monotonic on}(0,\infty),$$
$$\Leftrightarrow -f' \text{ is CM on}(0,\infty), \text{ and } \lim_{x\to\infty} f(x) \geq 0.$$

A positive function f is said to be logarithmically completely monotonic (LCM) on $(0,\infty)$ if $-(\log f)'$is completely monotonic (CM) on $(0,\infty)$ [[3], Definition 5.8]. Thus

$$f \text{ is LCM on}(0,\infty) \Leftrightarrow (-\log f(x))' \geq 0, (\log f)'' \text{ is CM on}(0,\infty),$$

$$\Leftrightarrow (\log f)'' \text{ is CM }, \text{ and } \lim_{x\to\infty}(-\log f(x))' \geq 0.$$

The class of logarithmically completely monotonic functions is a proper subset of the class of completely monotonic functions. Their importance stems from the fact that they represent Laplace transforms of innitely divisible probability distributions, see [[3], Thm. 5.9] and [[4], Sec. 51].

Euler gamma function is defined for positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

which is one of the most important special functions and has many extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

The logarithmic derivative of $\Gamma(x)$, denoted $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \geq 1$ are called the polygamma functions. The functions $\Gamma(x)$ and $\psi^{(k)}(x)$ for $k \geq 1$ are of fundamental importance in mathematics and have been extensively studied by many authors.

The $q-$analogue of Γ is defined by

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{j=0}^{\infty}\frac{1-q^{j+1}}{1-q^{j+x}},\ 0 < q < 1, \tag{1}$$

and

$$\Gamma_q(x) = (q-1)^{1-x}q^{\frac{x(x-1)}{2}}\prod_{j=0}^{\infty}\frac{1-q^{-(j+1)}}{1-q^{-(j+x)}},\ q > 1. \tag{2}$$

The $q-$gamma function $\Gamma_q(z)$ has the following basic properties:

$$\Gamma_q(z) = q^{\frac{(x-1)(x-2)}{2}}\Gamma_{\frac{1}{q}}(z). \tag{3}$$

and consequently

$$\log\Gamma_q(z) = \frac{x^2-3x+2}{2}\log(q) + \log\Gamma_{\frac{1}{q}}(z). \tag{4}$$

The $q-$digamma function ψ_q, the $q-$analogue of the psi or digamma function ψ is defined by

$$\psi_q(x) = \frac{\Gamma_q^{'}(x)}{\Gamma_q(z)} = -\log(1-q) + \log(q)\sum_{k=0}^{\infty}\frac{q^{k+x}}{1-q^{k+x}} \tag{5}$$

$$= -\log(1-q) + \log(q)\sum_{k=1}^{\infty}\frac{q^{kx}}{1-q^{k}},$$

for $0 < q < 1$, and from (2) we obtain for $q > 1$ and $x > 0$,

$$\psi_q(x) = -\log(1-q) + \log(q)\left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{q^{-(k+x)}}{1-q^{-(k+x)}}\right] \tag{6}$$

$$= -\log(1-q) + \log(q)\left[x - \frac{1}{2} - \sum_{k=1}^{\infty} \frac{q^{-kx}}{1-q^{-k}}\right].$$

In [5], the authors proved that the function $\psi_q(x)$ tends $\psi(x)$ on letting $q \to 1$.

An important fact for gamma function in applied mathematics as well as in probability is the Stirling's formula that gives a pretty accurate idea about the size of gamma function. With the Euler-Maclaurin formula, Moak [[6], p. 409] obtained the following $q-$analogue of Stirling's formula

$$\log\Gamma_q(x) \sim \left(x - \frac{1}{2}\right)\log\left(\frac{1-q^x}{1-q}\right) + \frac{\mathrm{Li}_2(1-q^x)}{\log q} + \frac{1}{2}H(q-1)\log q + C_{\hat{q}} \tag{7}$$

$$+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)$$

as $x \longrightarrow \infty$ where $H(.)$ denotes the Heaviside step function, $B_k,\ k = 1, 2, ...$ are the Bernoulli numbers,

$$\hat{q} = \begin{cases} q & \text{if } \ 0 < q < 1 \\ 1/q & \text{if } \ q > 1 \end{cases}$$

The function $\mathrm{Li}_2(z)$ is the dilogarithm function defined for complex argument z as

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt, \ z \notin [1, \infty) \tag{8}$$

P_k is a polynomial of degree k satisfying

$$P_k(z) = (z - z^2)P'_{k-1}(z) + (kz+1)P_{k-1}(z), \ P_0 = P_{-1} = 1, \ k = 1, 2, ... \tag{9}$$

and

$$C_q = \frac{1}{2}\log(2\pi) + \frac{1}{2}\log\left(\frac{q-1}{\log q}\right) - \frac{1}{24}\log q + \frac{1}{\log(q)}\int_0^{-\log(q)} \frac{udu}{e^u - 1}$$

$$+ \log\left(\sum_{m=-\infty}^{\infty} r^{m(6m+1)} - r^{(2m+1)(3m+1)}\right),$$

where $r = \exp(4\pi^2/\log q)$. In [6], the author proved the following formulas:

$$\lim_{q \longrightarrow 1} \frac{\mathrm{Li}_2(1-q^x)}{\log q} = -x, \ \text{ and } \ \lim_{q \to 1} c_q = \frac{1}{2}\log(2\pi). \tag{10}$$

Let α be a real number and q, β are nonnegative parameter. We define the function $f_{\alpha,\beta}(q;x)$ by [2]

$$f_{\alpha,\beta}(q;x) = \frac{\Gamma_q(x+\beta)\exp\left(\frac{-\operatorname{Li}_2(1-q^x)}{\log q}\right)}{\left(\frac{1-q^x}{1-q}\right)^{x+\beta-\alpha}},\ x>0. \tag{11}$$

It is worth mentioning that Chen and Qi [7] considered the function

$$f_{\alpha,\beta}(x) = \frac{e^x\Gamma(x+\beta)}{x^{x+\beta-\alpha}},\ x>0.$$

which is a special case of the function $f_{\alpha,\beta}(q;x)$ on letting $q \to 1$.

In [2], the author proved the following results:

Theorem A *Let $q \in (0,1)$ and α be a real number. The function $f_{\alpha,1}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, if and only if $2\alpha \leq 1$.*

Theorem B *Let $q \in (0,1)$ and α be a real number. The function $[f_{\alpha,1}(q;x)]^{-1}$ is logarithmically completely monotonic on $(0,\infty)$, if and only if $\alpha \geq 1$.*

Theorem C *Let $q \in (0,1)$ and α be a real number and $\beta \geq 0$. Then, the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic function on $(0,\infty)$ if $2\alpha \leq 1 \leq \beta$.*

Motivated by this results, our aim is to establish a sufficient condition, a necessary condition and a necessary and sufficient condition such that the function $f_{\alpha,\beta}(q;x)$ is logarithmic completely monotonic on $(0,\infty)$, when the real β is lies in different ranges. These results can be regarded as supplements to the paper [2]. As applications of these results, we derive the $q-$version of Kěckić-Vasić type inequality for $q>0$, and we find a necessary and sufficient condition for the function $g_{\alpha,\beta}(q;x)$ defined by

$$g_{\alpha,\beta}(q;x) = \log\Gamma_q(x+\beta) - \frac{\operatorname{Li}_2(1-q^x)}{\log(q)} - (x+\beta-\alpha)\log\left(\frac{1-q^x}{1-q}\right) \tag{12}$$

is completely monotonic on $(0,\infty)$.

As a tool for completing our work, we need to the following lemmas:

2 Lemmas

Lemma 2.1 *[6] The following approximation for the q−digamma function*

$$\psi_q(x) = \log\left(\frac{1-q^x}{1-q}\right) + \frac{1}{2}\frac{q^x \log(q)}{1-q^x} + O\left(\frac{q^x \log^2(q)}{(1-q^x)^2}\right), \tag{13}$$

holds for all $q > 0$ and $x > 0$.

Lemma 2.2 *[8] For every $x, q > 0$, there exists at least one real number $a \in [0, 1]$ such that*

$$\psi_q(x) = \log\left(\frac{1-q^{x+a}}{1-q}\right) + \frac{q^x \log(q)}{1-q^x} - \left(\frac{1}{2} - a\right) H(q-1)\log(q) \tag{14}$$

where $H(.)$ is the Heaviside step function.

Lemma 2.3 *The function*

$$g_1(x) = \frac{2\log(x) + x\log(x) - 2x + 2}{2\log(x)} \tag{15}$$

is decreasing on $(0, 1)$. *Furthermore, it satisfies* $\lim_{x\longrightarrow 0} g_1(x) = 1$ *and* $\lim_{x\longrightarrow 1} g_1(x) = 1/2$.

Proof. Differentiating $g(x)$ yields

$$g_1'(x) = \frac{h(x)}{4x\log^2(x)},$$

where $h(x) = x\log^2(x) - 2x\log(x) + 2x - 2$, so $h'(x) = \log^2(x) > 0$, for all $x \in (0, 1)$. Consequently, the function $h(x)$ is increasing on $(0, 1)$, such that $\lim_{x\longrightarrow 1} h(x) = 0$. Therefore the function $g_1(x)$ is decreasing on $(0, 1)$. It is easy to see that $\lim_{x\longrightarrow 0} g_1(x) = 1$, and by using the l'Hospital's rule we deduce that $\lim_{x\longrightarrow 1} g_1(x) = 1/2$.

Lemma 2.4 *The function*

$$g_2(x) = \frac{2x - x\log(x) - 2}{2\log(x)} \tag{16}$$

is increasing on $(0, 1)$. *Furthermore, it satisfies* $\lim_{x\longrightarrow 0} g_2(x) = 0$ *and* $\lim_{x\longrightarrow 1} g_2(x) = 1/2$.

Proof. We note that $g_2(x) = 1 - g_1(x)$. So, Lemma 2.3 completes the proof of Lemma 2.4.

3 Main results

We first state our main results as follows. The next Theorem is an extension of Theorem C.

Theorem 3.1 *Let $q > 0$, α be a real number and $\beta \geq 0$. If $2\alpha \leq 1 \leq \beta$, then the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$.*

Proof. Let $q > 1$, the relations (3), (4) and the definition of the $q-$digamma function (5) give

$$\psi_q(x+\beta) = \frac{2x+2\beta-3}{2}\log(q) + \psi_{1/q}(x+\beta). \tag{17}$$

A simple computation we get

$$\log\left(\frac{1-q^x}{1-q}\right) = \log\left(\frac{1-(1/q)^x}{1-(1/q)}\right) + (1-x)\log(1/q), \tag{18}$$

and

$$\frac{q^x}{1-q^x} = -q^x\frac{(1/q)^x}{1-(1/q)^x}. \tag{19}$$

By using the formula [[2], Lemma 1]

$$(\log(f_{\alpha,\beta}(q;x))' = \psi_q(x+\beta) - \log\left(\frac{1-q^x}{1-q}\right) + (\beta-\alpha)\frac{q^x\log(q)}{1-q^x} \tag{20}$$

and the previous formulas we get for $q > 1$

$$(\log(f_{\alpha,\beta}(q;x))' = \psi_{1/q}(x+\beta) - \log\left(\frac{1-(1/q)^x}{1-(1/q)}\right) + (\beta-\alpha)\frac{(1/q)^x\log(1/q)}{1-(1/q)^x} + (\alpha-1/2)\log(q) \tag{21}$$

By using the Theorem C, we deduce that the function $(-\log(f_{\alpha,\beta}(q;x))'$ is completely monotonic on $(0,\infty)$ for all $q \in (0,1)$ and $2\alpha \leq 1 \leq \beta$, and consequently the function

$$\begin{aligned}(-\log(f_{\alpha,\beta}(q;x))' - (1/2-\alpha)\log(q) &= \qquad (22)\\ = \log\left(\frac{1-(1/q)^x}{1-(1/q)}\right) - \psi_{1/q}(x+\beta) &+ (\alpha-\beta)\frac{(1/q)^x\log(1/q)}{1-(1/q)^x}\\ = \log\left(\frac{1-\hat{q}^x}{1-\hat{q}}\right) - \psi_{\hat{q}}(x+\beta) &+ (\alpha-\beta)\frac{\hat{q}^x\log(\hat{q})}{1-\hat{q}^x},\end{aligned}$$

is also completely monotonic on $(0,\infty)$ for all $q > 1$ and $2\alpha \leq 1 \leq \beta$. From this fact, and since

$$(1/2-\alpha)\log(q) \geq 0$$

for all $2\alpha \leq 1$ and $q > 1$, we deduce that the function $(-\log(f_{\alpha,\beta}(q;x))'$ is completely monotonic on $(0,\infty)$ for all $q > 1$ and $2\alpha \leq 1 \leq \beta$. Applying Theorem C with the above results, we obtain the desired results.

Remark 3.2 *A similar proof to proof of Theorem 3.1, we deduce that the Theorem A and Theorem B are valid for* $q > 0$, *and consequently, the inequalities proved in [2] are holds true for all* $q > 0$.

In the next Theorem we give a necessary condition for the Theorem C, where β is nonnegative.

Theorem 3.3 *For* $q \in (0,1), \alpha$ *be a real number and* $\beta > 0$. *If* $f_{\alpha,\beta}(q;x)$ *is logarithmically completely monotonic on* $(0,\infty)$, *then* $\alpha \le \min\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right)$.

Proof. Assume that the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, thus

$$(\log(f_{\alpha,\beta}(q;x)))' = \psi_q(x+\beta) - \log\left(\frac{1-q^x}{1-q}\right) + (\beta-\alpha)\frac{q^x\log(q)}{1-q^x} \le 0,$$

which is equivalent to

$$\beta - \alpha \ge \frac{1-q^x}{q^x\log(q)}\left(\log\left(\frac{1-q^x}{1-q}\right) - \psi_q(x+\beta)\right). \tag{23}$$

By means of Lemma 2.2 and (23) we obtain

$$\begin{aligned}\beta - \alpha &\ge \lim_{x\to 0}\left(\frac{1-q^x}{q^x\log(q)}\left[\log\left(\frac{1-q^x}{1-q}\right) - \psi_q(x+\beta)\right]\right)\\ &= \lim_{x\to 0}\left(\frac{1-q^x}{q^x\log(q)}\left[\log\left(\frac{1-q^x}{1-q^{x+a+\beta}}\right) - \frac{q^{x+\beta}\log(q)}{1-q^{x+\beta}}\right]\right) = 0.\end{aligned}$$

Now, letting $x \to \infty$ in (23) and using Lemma 2.1 we get

$$\begin{aligned}\beta - \alpha &\ge \lim_{x\to \infty}\left(\frac{1-q^x}{q^x\log(q)}\left[\log\left(\frac{1-q^x}{1-q}\right) - \psi_q(x+\beta)\right]\right)\\ &= \lim_{x\to \infty}\left(\frac{1-q^x}{q^x\log(q)}\left[\log\left(\frac{1-q^x}{1-q^{x+\beta}}\right) - \frac{q^{x+\beta}\log(q)}{2(1-q^{x+\beta})}\right]\right)\\ &= \frac{2q^\beta - q^\beta\log(q) - 2}{2\log(q)},\end{aligned}$$

which implies that

$$\alpha \le \beta - \frac{2q^\beta - q^\beta\log(q) - 2}{2\log(q)}.$$

Hence, the necessary condition such that the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, is given by

$$\alpha \le \min\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right).$$

Theorem 3.4 *Let $q \in (0,1)$, $\beta \in \{0\} \cup [1,\infty)$. Then the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, if and only if $\alpha \leq \frac{1}{2}$.*

Proof. Firstly, let $\beta \geq 1$, we know that the condition of Theorem 3.4 is sufficient, it is follows by Theorem C. Now, we prove the necessary condition. Suppose that the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$, then

$$\alpha \leq \min\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right), \tag{24}$$

by means of Theorem 3.3. As the function $\beta \mapsto k_q(\beta) = \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}$ is increasing on $[1,\infty)$, we have

$$k_q(\beta) \geq k_q(1) = g_2(q) \geq 0,$$

for $q \in (0,1)$, by Lemma 2.4. So, for $q \in (0,1)$, $\beta \geq 1$ we get

$$\min\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right) = \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}. \tag{25}$$

Now, for $q \in (0,1)$ and $\beta \geq 1$, we define the function $h(q;\beta)$ by

$$h(q;\beta) = \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}.$$

It is easily verified that the function $\beta \mapsto h(q;\beta)$ is strictly increasing and strictly convex on $[1,\infty)$ for each $q \in (0,1)$. From this fact and (24), we thus obtain

$$\alpha \leq h(q;1) = g(q),$$

where $q \in (0,1)$. From Lemma 2.3, we have

$$\alpha \leq 1/2.$$

Now, let $\beta = 0$. By the following relationship:

$$\psi_q(x+1) = \frac{1-q^x}{1-q}\psi_q(x) \tag{26}$$

and the definition of the function $f_{\alpha,\beta}(q;x)$ we deduce that

$$f_{\alpha,0}(q;x) = f_{\alpha,1}(q;x). \tag{27}$$

From this fact and Theorem A, we deduce the desired results for $\beta = 0$. So, the proof of Theorem 3.4 is evidently completed.

Remark 3.5 *Theorem 3.3 and Theorem 3.4, are shown to be a generalization of Theorem 1. (2) and (3), obtained by Guo et al. [1].*

As consequence of Theorem 3.1 and Theorem 3.4 we deduce the $q-$version of Kěckić-Vasić type inequality.

Corollary 3.6 *Let x, y be positive numbers with $x < y$, and $\beta \geq 1$.*
1. For $q > 0$, the following inequality

$$\left[\left[\frac{1-q^x}{1-q^y}\right]^{\alpha-\beta}\frac{\Gamma_q(x+\beta)}{\Gamma_q(x+\beta)}\right]^{\frac{\log(q)}{Li_2(1-q^y)-Li_2(1-q^x)}} \leq \frac{1}{e}\left[\frac{\left[\frac{1-q^x}{1-q}\right]^x}{\left[\frac{1-q^y}{1-q}\right]^y}\right]^{\frac{\log(q)}{Li_2(1-q^y)-Li_2(1-q^x)}} \tag{28}$$

holds true also if $\alpha \leq 1/2$.
2. For $q \in (0,1)$. The inequality (28) holds true if and only if $\alpha \leq 1/2$.

Proof. Since logarithmically completely monotonic function is completely monotonic function we conclude that the function $f_{\alpha,\beta}(q;x)$ is decreasing on $(0,\infty)$, and consequently we get

$$f_{\alpha,\beta}(q;x) \geq f_{\alpha,\beta}(q;y),$$

which is equivalent to (28). By considering the sufficient condition in Theorem 3.1 and the necessary and sufficient condition in Theorem 3.4.

Remark 3.7 *It is worth mentioning that the inequality (28) when letting q tends to 1, returns to the inequality (2) in [1].*

Theorem 3.8 *Let $q \in (0,1)$. Then the function $g_{\alpha,0}(q;x)$ is completely monotonic on $(0,\infty)$, if and only if $\alpha = 1/2$.*

Proof. It is clear that

$$g_{\alpha,\beta}(q;x) = \log f_{\alpha,\beta}(q;x). \tag{29}$$

Suppose that the function $g_{\alpha,\beta}(q;x)$ is completely monotonic on $(0,\infty)$. Hence, the function $f_{\alpha,\beta}(q;x)$ is logarithmically completely monotonic on $(0,\infty)$. By using Theorem 3.4 and (27), we deduce that

$$\alpha \leq 1/2. \tag{30}$$

On the other hand, by using the $q-$analogue of Stirling formula (7) and the definition of the function $g_{\alpha,\beta}(q;x)$ we obtain for $q \in (0,1)$

$$g_{\alpha,\beta}(q;x) \sim \left(\alpha-\frac{1}{2}\right)\log\left(\frac{1-q^x}{1-q}\right) + C_{\hat{q}} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(\frac{\log\hat{q}}{\hat{q}^x-1}\right)^{2k-1}\hat{q}^x P_{2k-3}(\hat{q}^x), \tag{31}$$

as $x \to \infty$. In view of the fact that $g_{\alpha,\beta}(q;x) \geq 0$, and (31) we get

$$\alpha - \frac{1}{2} \geq -\lim_{x\to\infty} \frac{C_{\hat{q}} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(\frac{\log \hat{q}}{\hat{q}^x - 1}\right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)}{\log\left(\frac{1-q^x}{1-q}\right)} = 0,$$

that is,

$$\alpha \geq 1/2. \tag{32}$$

Combining (30) and (32) we obtain

$$\alpha = 1/2.$$

Conversely, In view of (29) and using the fact that the function $f_{1/2,0}(q;x)$ is logarithmic completely monotonic on $(0,\infty)$, (see Theorem 3.4) we have

$$(-1)^n g_{1/2,0}^{(n)}(q;x) = (-1)^n (\log f_{1/2,0}(q;x))^{(n)} \geq 0, \tag{33}$$

for all $n \geq 1$. So, the function $g_{1/2,0}(q;x)$ is decreasing on $(0,\infty)$. Thus

$$g_{1/2,0}(q;x) \geq \lim_{x\to\infty} g_{1/2,0}(q;x). \tag{34}$$

By (31) and (34), we conclude that

$$g_{\alpha,0}(q;x) \geq \lim_{x\to\infty} g_{1/2,0}(q;x) = C_{\hat{q}} > 0,$$

from which we readily see that (33) is also valid for $n = 0$. Consequently, the function $g_{1/2,0}(q;x)$ is completely monotonic on $(0,\infty)$.

Theorem 3.9 *For $q \in (0,1)$, α be a real number and $\beta > 0$. If $(f_{\alpha,\beta}(q;x))^{-1}$ is logarithmically completely monotonic on $(0,\infty)$, then $\alpha \geq \max\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right)$.*

Proof. The proof of this Theorem is similar of the proof of Theorem 3.3.

Corollary 3.10 *Let $q \in (0,1)$, $\beta \geq 1$. If the function $(f_{\alpha,\beta}(q;x))^{-1}$ is logarithmically completely monotonic on $(0,\infty)$, then $\alpha \geq \beta$.*

Proof. Suppose that the function $(f_{\alpha,\beta}(q;x))^{-1}$ is logarithmically completely monotonic on $(0,\infty)$, then

$$\alpha \geq \max\left(\beta, \beta - \frac{2q^\beta - \log(q)q^\beta - 2}{2\log(q)}\right),$$

by Theorem 3.10. From the previous inequality and (25) we deduce

$$\alpha \geq \beta$$

for all $\beta \geq 1$ and $q \in (0,1)$.

Remark 3.11 *Let $\beta = 1$ in the precedent Corollary we obtain the necessary condition of Theorem B.*

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