Super-biderivations of the super Virasoro algebra

Jinsen Zhou

School of Information Engineering, Longyan University, Longyan 364012, Fujian, P. R. China
Email: zjs9932@126.com

Guangzhe Fan

School of Mathematical Sciences, Tongji University, Shanghai 200092, P. R. China
Email: yzfanguangzhe@126.com

Abstract

In this paper we investigate super-biderivations of the super Virasoro algebra. The super Virasoro algebra is a Lie superalgebra equipped with a basis \( \{L_m, I_m, G_m \mid m \in \mathbb{Z} \} \) and nontrivial Lie super-brackets:

\[
[L_m, L_n] = (n - m)L_{m+n}, \quad [L_m, I_n] = nI_{m+n}, \quad [L_m, G_n] = (n - m)G_{m+n},
[I_m, G_n] = G_{m+n}.
\]

Finally, we prove that all skew-supersymmetric super-biderivations of the super Virasoro algebra are inner super-biderivations.

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1 Introduction

Recently, many authors have investigated biderivations of both Lie algebras and Lie superalgebras. In [3,7], the authors proved that each skew-symmetric biderivation on the Schrödinger-Virasoro algebra and a simple generalized Witt algebra over a field of characteristic 0 is a inner biderivation. Later on, super-biderivations of many Lie superalgebras were studied in [4,9,10]. In [5], the author introduced the concept of the super Virasoro algebra. In [11], the
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authors studied Lie super-bialgebra and quantization of the super Virasoro algebra.

The super Virasoro algebra $S$ is a Lie superalgebra whose even part $S_0$ has a basis $\{L_m, I_m \mid m \in \mathbb{Z}\}$ and odd part $S_1$ has a basis $\{G_m \mid m \in \mathbb{Z}\}$, equipped with the following nontrivial Lie super-brackets $(m, n \in \mathbb{Z})$: 
\[
\begin{align*}
[L_m, L_n] &= (n - m)L_{m+n}, &[L_m, I_n] = nI_{m+n}, &[L_m, G_n] = (n - m)G_{m+n}, &[I_m, G_n] = G_{m+n}.
\end{align*}
\]

Obviously, we know that $S$ contains many important subalgebras. For example,

- $W = \oplus_{m \in \mathbb{Z}} L_m$ is in fact the well-known centerless Virasoro algebra.

- $H = (\oplus_{m \in \mathbb{Z}} L_m) \oplus (\oplus_{m \in \mathbb{Z}} I_m)$ is the centerless twisted Heisenberg-Virasoro algebra.

Note that $S$ is $\mathbb{Z}$-graded: $S = \oplus_{m \in \mathbb{Z}} S_m, S_m = \text{span}\{L_m, I_m, G_m\}$.

Here is a detailed outline of the contents of the main parts of the article. In Section 2, we review some conclusions about super-biderivations of Lie superalgebras. In Section 3, we prove that all skew-supersymmetric super-biderivations of the super Virasoro algebras are inner super-biderivations.

For the readers’ convenience, we give some notations used in this paper. Denote by $C$, $\mathbb{Z}$ the sets of complex numbers, integers. We assume that all vector spaces are based on $C$ and the degree of $x$ or $\phi$ is denoted by $|x|$ or $|\phi|$. In addition, $x$ is always assumed to be homogeneous when $|x|$ occurs. Denote by $hg(L)$ the set of all homogeneous elements of $L$ where $L$ be a superspace.

2 Preliminaries

In this section, we shall summarize some basic concepts about super-biderivations of Lie superalgebras in [4,9].

**Definition 2.1** We call a bilinear map $\phi: L \times L \to L$ a super-biderivation of $L$ if it satisfies the following two equations $(x, y, z \in hg(L))$:
\[
\begin{align*}
\phi([x, y], z) &= (-1)^{|\phi||x|}[x, \phi(y, z)] + (-1)^{|y||z|}|\phi(x, z), y], \quad (1) \\
\phi(x, [y, z]) &= [\phi(x, y), z] + (-1)^{|\phi||x|+|y|}|y, \phi(x, z)]. \quad (2)
\end{align*}
\]

**Proposition 2.2** We say that a super-biderivation $\phi$ of $L$ is a skew-supersymmetric super-biderivation if $\phi$ satisfies the following condition $(x, y \in hg(L))$:
\[
\text{skew-supersymmetry : } \phi(x, y) = -(-1)^{|x||y|}\phi(y, x). \quad (3)
\]
Definition 2.3 A super-biderivation $\phi$ of homogenous $\gamma \in Z_2$ of $L$ is a super-biderivation such that $\phi(L_{\alpha}, L_{\beta}) \subseteq L_{\alpha+\beta+\gamma}$ for any $\alpha, \beta \in Z_2$. Denote by $\text{BDer}_{\gamma}(L)$ the set of all super-biderivations of homogenous $\gamma$ of $L$. Obviously, $\text{BDer}(L) = \text{BDer}_0(L) \oplus \text{BDer}_1(L)$.

Lemma 2.4 If the map $\phi_\lambda: L \times L \rightarrow L$, defined by $\phi_\lambda(x, y) = \lambda[x, y]$ for any $x, y \in hg(L)$, where $\lambda \in C$, then $\phi_\lambda$ is a skew-supersymmetric super-biderivation of $L$. We call this class super-biderivations inner super-biderivations.

Lemma 2.5 Let $\phi$ be a skew-supersymmetric super-biderivation on $L$, then we get $[\phi(x, y), [u, v]] = (-1)^{|\phi||[x, y]|}[[x, y], \phi(u, v)]$ for any $x, y, u, v \in hg(L)$.

Lemma 2.6 Let $\phi$ be a skew-supersymmetric super-biderivation on $L$. If $|x| + |y| = 0$, then $[\phi(x, y), [x, y]] = 0$ for any $x, y \in hg(L)$.

Lemma 2.7 Let $\phi$ be a skew-supersymmetric super-biderivation on $L$. If $[x, y] = 0$, then $\phi(x, y) \in Z([L, L])$, where $Z([L, L])$ is the center of $[L, L]$.

3 Super-biderivations of the super Virasoro algebra

In this section, we would like to compute super-biderivations of the super Virasoro algebras.

Lemma 3.1 Every super-skewsymmetric super-biderivation on the super Virasoro algebra $S$ is an inner super-biderivation.

Proof Suppose $\phi$ is a super-biderivation of the super Virasoro algebra $S$. Assume that $\phi(L_0, L_n) = \Sigma_{m \in Z}(a^n_m L_m + b^n_m I_m + c^n_m G_m)$, $\phi(L_0, I_n) = \Sigma_{m \in Z}(d^n_m L_m + e^n_m I_m + f^n_m G_m)$, $\phi(L_0, G_n) = \Sigma_{m \in Z}(g^n_m L_m + h^n_m I_m + \Delta^n_m G_m)$, where $a^n_m, b^n_m, c^n_m, d^n_m, e^n_m, f^n_m, g^n_m, h^n_m, \Delta^n_m \in C$ for any $m, n \in Z$.

According to Lemma 2.7, then $\phi(L_0, L_0), \phi(L_0, I_0), \phi(L_0, G_0) \in Z([S, S])$. Hence, $\phi(L_0, L_0) = \phi(L_0, I_0) = \phi(L_0, G_0) = 0$.

Due to $L_m \in S_0$, then $|L_m| + |L_n| = 0$ for any $m, n \in Z$. By Lemma 2.6, then we obtain

$$[[L_0, L_n], \phi(L_0, L_n)] = 0.$$ 

Furthermore,

$$n[L_n, \Sigma_{m \in Z}(a^n_m L_m + b^n_m I_m + c^n_m G_m)] = 0.$$ 

One has

$$a^n_m(m - n) = b^n_m m = c^n_m(m - n) = 0.$$
Thus, \( a^n_m = c^n_m = 0 \) for \( m \neq n \) and \( b^n_m = 0 \) for \( m \neq 0 \). So we get
\[
\phi(L_0, L_n) = a^n_n L_n + b^n_0 I_0 + c^n_n G_n.
\]

By Lemma 2.5, we have
\[
[\phi(L_0, L_n), [L_0, L_1]] = (-1)^{\phi([L_0]+[L_n])}[[L_0, L_n], \phi(L_0, L_1)].
\]

Hence, we deduce that \( a^n_n = n a^1_1 \) and \( c^n_n = n c^1_1 \).

Let \( \lambda = a^1_1, \mu = c^1_1 \), then we have
\[
\phi(L_0, L_n) = \lambda n L_n + b^n_0 I_0 + \mu n G_n.
\]

By Lemma 2.5, we have
\[
[\phi(L_0, L_k), [L_0, I_n]] = (-1)^{\phi([L_0]+[I_n])}[[L_0, L_k], \phi(L_0, I_n)]
\].

One deduces that
\[
\phi(L_0, L_n) = \lambda n L_n + b^n_0 I_0,
\]
\[
\phi(L_0, I_n) = \lambda n G_n + d^n_0 I_0.
\]

Set \( x = L_0, y = I_0, z = G_n \) in (2), then
\[
\phi(L_0, [I_0, G_n]) = [\phi(L_0, I_0), G_n] + [I_0, \phi(L_0, G_n)].
\]

Hence, we have \( \phi(L_0, G_n) = \Sigma_{m \in Z} (\triangle^n_m G_m) \).

Set \( x = L_0, y = L_0, z = G_n \) in (2), then
\[
\phi(L_0, [L_0, G_n]) = [\phi(L_0, L_0), G_n] + [L_0, \phi(L_0, G_n)].
\]

Hence, we have \( \phi(L_0, G_n) = \triangle^n_n G_n \).

Set \( x = L_0, y = I_k, z = G_n \) in (2), then we have
\[
\phi(L_0, [I_k, G_n]) = [\phi(L_0, I_k), G_n] + [I_k, \phi(L_0, G_n)].
\]

This shows that \( \triangle^n_n = \lambda n \) and \( b^n_0 = 0 \).

Set \( x = L_0, y = I_k, z = G_n \) in (2), then we have
\[
\phi(L_0, [I_k, G_n]) = [\phi(L_0, I_k), G_n] + [I_k, \phi(L_0, G_n)].
\]

Therefore, we have \( d^n_0 = 0 \).

Finally, we have proved the following equations \( (n \in Z) \):
\[
\phi(L_0, L_n) = \lambda [L_0, L_n],
\]
\[
\phi(L_0, I_n) = \lambda [L_0, I_n],
\]
\[
\phi(L_0, G_n) = \lambda [L_0, G_n].
\]
For any $z \in S$, we get
$$\phi(L_0, z) = \lambda[L_0, z].$$

Due to $|\phi| = 0$, and according to Lemma 2.5, we obtain
$$[\phi(x, y), [L_0, z]] = [[x, y], \phi(L_0, z)].$$

Furthermore we have $[\phi(x, y) - \lambda[x, y], [L_0, z]] = 0$. According to the arbitrary of $z$, then $\phi(x, y) - \lambda[x, y] = 0$.
Thus, $\phi(x, y) = \lambda[x, y]$.

References


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