Structures of $n$-Lie algebra $A^n$

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Abstract

In this paper, we discuss the structure of the exterior direct sum $n$-Lie algebra $(A^n, [\cdot, \cdot, \cdot, [\cdot]])$ of an $n$-Lie algebra $A$. And it is proved that, (1) if $I_1, \cdots, I_{n-1}$ are ideals of an $n$-Lie algebra $A$, then the vector space $(I_1, I_2, \cdots, I_{k-1}, I_1, I_{k+1}, \cdots, I_{n-1})$ is also an ideal of $(A^n, [\cdot, \cdot, \cdot, [\cdot]])$, and if $I$ is a solvable (nilpotent) ideal of $A$, then $I^n$ is also solvable (nilpotent). (2) For a linear mapping $\delta \in \text{End}(A)$, then $\delta$ is a derivation of $A$ if and only if $f_\delta \in \text{Hom}(A, A^n)$ is an $n$-Lie algebra homomorphism. (3) If $(V, \rho)$ is an $A$-module, then $(V^n, \bar{\rho})$ is an $A^n$-module.

2010 Mathematics Subject Classification: 17B05 17D30
Keywords: $n$-Lie algebra, exterior direct sum $n$-Lie algebra, derivation, module.
1 Preliminary

In the paper [1], authors provided the exterior direct sum n-Lie algebras of n-Lie algebras [2, 3]. In this paper, we mainly study the structures of the exterior direct sum n-Lie algebra of a given n-Lie algebra. First, we recall some notions. Let $A$ be a vector space. The direct sum vector space of $A$ is $A^n = \{(x_1, \cdots, x_n) \mid x_i \in A, 1 \leq i \leq n\}$, satisfying that for all $X = (x_1, \cdots, x_n)$ and $Y = (y_1, \cdots, y_n) \in A^n$ and $\lambda \in F$,

$$X + Y = (x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n),$$

$$\lambda X = \lambda (x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n).$$

An $n$-Lie algebra [3] is a vector space $A$ over a field $F$ endowed with an $n$-ary multilinear skew-symmetric multiplication satisfying that for all $x_1, \cdots, x_n, y_2, \cdots, y_{n-1} \in A$,

$$[[x_1, \cdots, x_n], y_2, \cdots, y_n] = \sum_{i=1}^{n} [x_1, \cdots, [x_i, y_2, \cdots, y_n], \cdots, x_n]. \quad (1)$$

The identity (1) is usually called the n-Jacobi identity.

Let $A$ be an $n$-Lie algebra. A derivation of an $n$-Lie algebra $A$ is a linear mapping $D : A \rightarrow A$ satisfying that

$$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, D(x_i), \cdots, x_n], \quad \forall x_1, \cdots, x_n \in A.$$

By Eq.(1), for $x_1, \cdots, x_{n-1} \in A$, the left multiplication $ad(x_1, \cdots, x_{n-1}) : A \rightarrow A$ defined by for all $x \in A$, $ad(x_1, \cdots, x_{n-1}, x) = [x_1, \cdots, x_{n-1}, x]$ is a derivation of $A$. All the derivations of $A$, denoted by $Der(A)$, is a subalgebra of the general linear algebra $gl(A)$.

Let $A$ be an $n$-Lie algebra and $V$ be a vector space. If there exists a linear mapping $\rho : A^{\wedge(n-1)} \rightarrow End(V)$ satisfying that for all $x_i, y_i \in A, i = 1, \cdots, n$,

$$\rho([x_1, \cdots, x_n], y_2, \cdots, y_{n-1})$$

$$= \sum_{i=1}^{n} (-1)^{n-i} \rho(x_1, \cdots, \hat{x_i}, \cdots, x_n) \rho(x_i, y_2, \cdots, y_{n-1}), \quad (2)$$

$$[\rho(x_1, \cdots, x_{n-1}), \rho(y_1, \cdots, y_{n-1})] = \sum_{i=1}^{n} \rho(y_1, \cdots, [x_1, \cdots, x_{n-1}, y_i], \cdots, y_{n-1}) \quad (3)$$

then $(V, \rho)$ is called a representation of $A$, or $V$ is an $A$-module [4].

As an example, the linear mapping $\rho : A^{\wedge2} \rightarrow End(A)$ defined by for all $x_1, \cdots, x_{n-1} \in A$, $\rho(x_1, \cdots, x_{n-1}) = ad(x_1, \cdots, x_{n-1})$, $(A, ad)$ is an $A$-module, which is called the adjoint module of $A$.

Let $A$ be an $n$-Lie algebra and $V$ be a subspace of $A$. If $V$ satisfies that $[V, \cdots, V] \subseteq V$, then $V$ is a subalgebra of the $n$-Lie algebra $A$. If $V$ satisfies that $[V, A, \cdots, A] \subseteq V$, then $V$ is called an ideal of the $n$-Lie algebra $A$. If $V$ satisfies that $[V, \cdots, V] = 0$ ( $[V, V, A, \cdots, A] = 0$ ), then $V$ is called an abelian subalgebra (an abelian ideal).
2 Structures of $n$-Lie algebra $A^n$

**Lemma 2.1** Let $A$ be an $n$-Lie algebra. Then for any $s \geq 2$, $A^n$ is an $n$-Lie algebra in the multiplication $[\cdots,\cdot,\cdots]_s$, where for all $X_j = (x_1^j, \cdots, x_n^j) \in A^n$, $j = 1, \cdots, n$,

$$[X_1, \cdots, X_n]_s = \left( \sum_{i=1}^{n} [x_1^i, \cdots, x_1^n, [x_2^i, \cdots, x_2^n, \cdots, [x_n^i, \cdots, x_n^n]]_s \right).$$

The $n$-Lie algebra $(A^n, [\cdots,\cdot,\cdots])$ is called the exterior direct sum $n$-Lie algebra. For the similarity, in the following, we mainly discuss the case $s = 2$.

**Theorem 2.2** Let $A$ be an $n$-Lie algebra, $I_i, i = 1, \cdots, n-1$ be ideals of $A$. Then

$$U = (I_1, I_1, I_2, \cdots, I_{n-1}) = \{(x_1, \cdots, x_n) \mid x_1, x_2 \in I_1, x_i \in I_i, 3 \leq i \leq n\}$$

is an ideal of $(A^n, [\cdots,\cdot,\cdots])$.

**Proof** For all $(y_1, \cdots, y_n) \in U$, $x_i \in A, 1 \leq i \leq n$, $2 \leq j \leq n$, by Eq.(4),

$$[(y_1, \cdots, y_n), (x_1^2, \cdots, x_n^2), (x_1^3, \cdots, x_n^3)]_2 = ([y_1, x_2^2, \cdots, x_n^2], [y_2, x_2^2, \cdots, x_2^2], \cdots, [y_n, x_n^2, \cdots, x_n^2]) + (\sum_{i=2}^{n} [y_1, \cdots, x_i^2, \cdots, x_n^2, y_2, x_2^2, \cdots, x_2^2, \cdots, [y_n, x_n^2, \cdots, x_n^2]).$$

Since $I_j$ for $j = 1, \cdots, n-1$ are ideals of $A$ and $y_1, y_2 \in I_1$, we obtain that $[(y_1, \cdots, y_n), (x_1^2, \cdots, x_n^2), (x_1^3, \cdots, x_n^3)]_2 \in U$. It follows the result.

**Theorem 2.3** Let $A$ be an $n$-Lie algebra, $I_1, \cdots, I_{n-1}$ be ideals of the $n$-Lie algebra $A$. Then for any $3 \leq k \leq n$, $U_k = (I_1, I_2, \cdots, I_{k-1}, I_k, I_{k+1}, \cdots, I_{n-1})$ is an ideal of the $n$-Lie algebra $(A_n, [\cdots,\cdot,\cdots]).$

**Proof** The proof is similar to Theorem 2.2.

**Theorem 2.4** Let $A$ be an $n$-Lie algebra, $I$ be a solvable (nilpotent) ideal of $A$. Then $I^r$ is a solvable (nilpotent) ideal of the $n$-Lie algebras $(A^n, [\cdots,\cdot,\cdots])$, $2 \leq k \leq n$. Especially, if $I$ is an abelian ideal of the $n$-Lie algebra $A$, then $W = (I, \cdots, I)$ is an abelian ideal.

**Proof** By Theorem 2.2, $I^n$ is an ideal of $n$-Lie algebras $(A^n, [\cdots,\cdot,\cdots])$, $3 \leq k \leq n$. We only need to prove the solvability and the nilpotency. Since the similarity, we only prove the case $k = 2$. Denote $W = (I, \cdots, I)$. By hypothesis, there exists a number $s \geq 0$ such that $I^{(s)} = 0$. We will show that for any $r \geq 0$, $W^{(r)} \subseteq (I^{(r)}, \cdots, I^{(r)})$.

For all $y_i^l \in I$, $x_i^l \in A, 1 \leq l \leq 2; 3 \leq j \leq n; 1 \leq i \leq n$, suppose

$$[(y_1^1, \cdots, y_n^1), (y_2^1, \cdots, x_n^2), (x_1^3, \cdots, x_n^3), \cdots, (x_1^n, \cdots, x_n^n)]_2 = (z_1, \cdots, z_n).$$

Thanks to Eq.(4), for $2 \leq t \leq n$, $z_t = [y_1^1, y_1^2, x_1^3, \cdots, x_t^2] \in I^{(1)}$, and

$$z_1 = [y_1^1, y_2^2, x_1^3, \cdots, x_2^2] + [y_2^1, y_1^2, x_2^3, \cdots, x_2^2] + \sum_{l=3}^{n} [y_1^l, y_2^l, x_2^2, \cdots, x_2^2] \in I^{(1)}.$$
Therefore, \( z_i \in I^{(1)} \) for \( 1 \leq t \leq n \), we obtain \( W^{(1)} \subseteq (I^{(1)}, \cdots, I^{(1)}) \).

Now suppose \( W^{(s-1)} \subseteq (I^{(s-1)}, \cdots, I^{(s-1)}) \). By Theorem 2.2 and a similar discussion, we have
\[
W^{(s)} = [W^{(s-1)}, W^{(s-1)}, A^n, \cdots, A^n]_2 \subseteq (I^{(s)}, \cdots, I^{(s)}).
\]
Since \( I^{(s)} = 0 \), we have \( W^{(s)} = 0 \), that is, \( W \) is solvable.

Similar discussion, if \( I \) is nilpotent, then \( W \) is a nilpotent ideal of the exterior direct sum \( n \)-Lie algebra \( A^n \).

If \( I \) is an abelian ideal, then \( [I, I, A, \cdots, A] = 0 \). Then for all \( X_i = (x_i^1, \cdots, x_i^n) \in A^n \), \( 1 \leq i \leq n \), where \( X_1, X_2 \in I^n \), by Eq.(4), \([X_1, X_2, X_3, \cdots, X_n]_2 = 0\). Therefore, \( I^n \) is an abelian ideal. The proof is complete.

Now we discuss the relation between derivations of \( A \) with \((A^n, [\cdot, \cdots, \cdot]_k)\), for \( k \geq 3 \). Since the similarity of the discussion, we only study the case \( k = 2 \).

For convenience, in the following the exterior direct sum \( n \)-Lie algebra \((A^n, [\cdot, \cdots, \cdot], _2)\) of an \( n \)-Lie algebra \( A \) is simply denoted by \( A_n \).

**Theorem 2.5** Let \( A \) be an \( n \)-Lie algebra, \( \delta \in \text{End}(A) \). Define linear mapping \( f_\delta : A \to A^n \) by the formula
\[
f_\delta(x) = (\delta x, x, \cdots, x), \forall x \in A.
\] (5)

Then \( \delta \) is a derivation of \( A \) if and only if \( f_\delta \) is an algebra homomorphism.

**Proof** If \( \delta \) is a derivation of \( A \). Then for all \( x_i \in A, i = 1, \cdots, n \), by Eq.(4) and Eq.(5),
\[
f_\delta([x_1, x_2, \cdots, x_n]) = \delta([x_1, x_2, \cdots, x_n]), [x_1, \cdots, x_n], \cdots, [x_1, \cdots, x_n]),
\]
\[
[\delta(x_1), \cdots, f_\delta(x_n)]_2
\]
\[
= ([\delta(x_1), \cdots, x_1], (\delta(x_2), \cdots, x_2), \cdots, (\delta(x_n), \cdots, x_n)]_2
\]
\[
= (\sum_{i=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n], [x_1, \cdots, x_n], \cdots, [x_1, \cdots, x_n]).
\]
Since \( \delta([x_1, x_2, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n] \), we have
\[
f_\delta([x_1, x_2, \cdots, x_n]) = [\delta(x_1), f_\delta(x_2)]_2.
\]

Conversely, if \( f_\delta \) is an \( n \)-Lie algebra homomorphism, then for all \( x_i \in A, 1 \leq i \leq n \), \( f_\delta([x_1, \cdots, x_n]) = f_\delta([x_1, x_2, x_3, \cdots, x_n]) \). Thanks to Eq.(4) and Eq.(5),
\[
[\delta(x_1), \cdots, x_1], (\delta(x_2), \cdots, x_2), \cdots, (\delta(x_n), \cdots, x_n)]_2
\]
\[
= (\sum_{i=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n], [x_1, x_2, \cdots, x_n], \cdots, [x_1, x_2, \cdots, x_n])
\]
\[
= (\delta([x_1, x_2, \cdots, x_n]), \cdots, [x_1, x_2, \cdots, x_n]).
\]
Therefore, \( \delta([x_1, x_2, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, \delta(x_i), \cdots, x_n] \), that is, \( \delta \) is a derivation of \( A \). The proof is complete.

At last of the paper, we study the representation of the exterior direct sum \( n \)-Lie algebras. Let \( A \) be an \( n \)-Lie algebra, \( V \) be a vector space and \( \rho : A^{n-1} \to \text{End}(V) \) be a linear mapping. By the paper [4], \( (V, \rho) \) is an \( n \)-Lie algebra \( A \)-module if and only if the direct sum vector space \( B = A \oplus V \) is an
$n$-Lie algebra in the following multiplication, for all $x_i \in A, v \in V, 1 \leq i \leq n,$

$$[x_1, \cdots, x_n]_B = [x_1, \cdots, x_n], \quad [x_1, \cdots, x_{n-1}, v]_B = \rho(x_1, \cdots, x_{n-1})v,$$

and $V$ is an abelian ideal, that is, $[A, \cdots, A, V, V]_B = 0.$ Then we have the following result.

**Theorem 2.6** Let $A$ be an $n$-Lie algebra, $(V, \rho)$ be a representation of $n$-Lie algebra $A$. Then $(V^n, \bar{\rho})$ is a representation of the exterior direct sum $n$-Lie algebra $A^n$, where the linear mapping $\bar{\rho} : (A^n)^{\wedge n-1} \to \text{End}(V^n)$ defined by for all $X_i = (x_1^i, \cdots, x_n^i) \in A^n, 1 \leq i \leq n-1,$ and $u = (u_1, \cdots, u_n) \in V^n,$

$$\bar{\rho}(X_1, \cdots, X_{n-1})u = (\sum_{i=1}^{n-1} \rho(x^i_2, \cdots, x^i_1, x^i_{n-1})u_2 + \rho(x^1_2, \cdots, x^{n-1}_2)u_1, \rho(x^2_1, \cdots, x^{n-1}_1)u_2, \cdots, \rho(x^n_1, \cdots, x^1_1)u_n).$$

**Proof** Since $(V, \rho)$ is a representation of $A,$ then $(B = A \oplus V, [\cdots, \cdots, ]_B)$ is an $n$-Lie algebra. Therefore, we obtain the exterior direct sum $n$-Lie algebra $(B^n, [\cdots, \cdots, ]_B)$ of the $n$-Lie algebra $(B = A \oplus V, [\cdots, \cdots, ]_B).$ From $V$ is an abelian ideal of $B,$ and Theorem 2.2, $V^n$ is an abelian ideal of the $n$-Lie algebra $(B^n, [\cdots, \cdots, ]_B).$

Define linear mapping $\bar{\rho} : (A^n)^{\wedge n-1} \to \text{End}(V^n)$ by for all $X_1, \cdots, X_{n-1} \in A^n, w = (w_1, \cdots, w_n) \in V^n,$

$$\bar{\rho}(X_1, \cdots, X_{n-1})(w) = \text{ad}_{B^n}(X_1, \cdots, X_{n-1})(w) = [X_1, \cdots, X_{n-1}, w]_2.$$  

By a direct computation, $\bar{\rho}$ satisfies Eq.(2) and Eq.(3). Therefore, $(V^n, \bar{\rho})$ is a representation of $A^n.$ The proof is complete.

**Acknowledgements**

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

**References**


Received: August 31, 2016