Abstract
The paper main concerns the structure of 8-dimensional 3-Lie algebra $J_{11}$ which is constructed by 2-cubic matrix. The multiplication of $J_{11}$ is discussed and the decomposition of $J_{11}$ associate with a Cartan subalgebra is provided. The structure of derivation algebra and inner derivation algebra of $J_{11}$ are also studied.

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1 Introduction
$n$-Lie algebras [1-2], especially, 3-Lie algebras, have wide applications in mathematics and mathematical physics [3-4]. Researchers try to construct $n$-Lie algebras by algebras which we know well. For example, by means of one and two dimensional extensions, people constructed $n$-Lie algebras from $(n − 1)$-Lie algebras. In papers [5-6], 3-Lie algebras are constructed by Lie algebras, associative algebras, pre-Lie algebras and commutative associative algebras and their derivations and involutions. In paper [7], fifteen kinds of multiplications of $N$-cubic matrix are provided, and four non-isomorphic $N^3$-dimensional 3-Lie algebras are constructed. In this paper, we pay our main attention to
8-dimensional 3-Lie algebras which are constructed by 2-cubic matrix, we suppose that 3-Lie algebras over a field \( F \) of characteristic of zero, and the subspace generated by a subset \( S \) of a vector space \( V \) is denoted by \( < S > \).

2 Structure of 3-Lie algebras \( J_{11} \)

An \( N \)-order cubic matrix \( A = (a_{ijk}) \) (see [7]) over a field \( F \) is an ordered object which the elements with 3 indices, and the element in the position \((i, j, k)\) is \((A)_{ijk} = a_{ijk}, 1 \leq i, j, k \leq N \). Denote the set of all cubic matrix over a field \( F \) by \( \Omega \). Then \( \Omega \) is an \( N^3 \)-dimensional vector space over \( F \) with \( A + B = (a_{ijk} + b_{ijk}) \in \Omega, \lambda A = (\lambda a_{ijk}) \in \Omega \), for \( \forall A = (a_{ijk}), B = (b_{ijk}) \in \Omega, \lambda \in F \), that is, \((A + B)_{ijk} = a_{ijk} + b_{ijk}, (\lambda A)_{ijk} = \lambda a_{ijk} \).

Denote \( E_{ijk} \) a cubic matrix with the element in the position \((i, j, k)\) is 1 and elsewhere are zero. Then \( \{E_{ijk}, 1 \leq i, j, k \leq N\} \) is a basis of \( \Omega \), and for every \( A = (a_{ijk}) \in \Omega \), \( A = \sum_{1 \leq i, j, k \leq N} a_{ijk}E_{ijk}, a_{ijk} \in F \).

For all \( A = (a_{ijk}), B = (b_{ijk}) \in \Omega \), define the multiplication \( *_{11} \) in \( \Omega \) by

\[
(A *_{11} B)_{ijk} = \sum_{p=1}^{N} a_{ijp}b_{ipk},
\]

then \( (\Omega, *_{11}) \) is associative algebra.

Denote \( \langle A \rangle_1 = \sum_{p=1}^{N} a_{ppq} \). Then \( \langle \rangle_1 \) is linear functions from \( \Omega \) to \( F \) and satisfies \( \langle A *_{11} B \rangle_1 = \langle B *_{11} A \rangle_1 \).

Define the multiplication \( [\cdot, \cdot]_{11} : \Omega \land \Omega \land \Omega \rightarrow \Omega \) as follows:

\[
[A, B, C]_{11} = \langle A \rangle_1 (B *_{11} C - C *_{11} B) + \langle B \rangle_1 (C *_{11} A - A *_{11} C) + \langle C \rangle_1 (A *_{11} B - B *_{11} A).
\]

We obtain the following lemma.

**Theorem 2.1** The linear space \( \Omega \) is a 3-Lie algebra in the multiplication \( [\cdot, \cdot]_{11} \), which is denoted by \( J_{11} \).

In the following we suppose \( N = 2 \). We have the following result.

**Theorem 2.2** The 3-Lie algebra \( J_{11} \) is a non-nilpotent indecomposable 3-Lie algebra with a basis \( e_1 = E_{111}, e_2 = E_{112}, e_3 = E_{121}, e_4 = E_{111} - E_{122}, e_5 = E_{211} - E_{111}, e_6 = E_{212}, e_7 = E_{221}, e_8 = E_{211} - E_{222} \), and the multiplication in it is as follows:

\[
\begin{align*}
[e_1, e_6, e_7] &= e_8, [e_1, e_6, e_8] = -2e_6, [e_1, e_7, e_8] = 2e_7, \\
\end{align*}
\]

Then center of \( J_{11} \) is \( < e_4 + 2e_5 - e_8 > \).

**Proof** It is clear that \( \{e_1, \ldots, e_8\} \) is a basis of \( \Omega \). By the definition of \( [\cdot, \cdot]_{11} \), we obtain Eq.(2). Thank to \( ad(e_1, e_4) \) is non-nilpotent, the 3-Lie algebra \( J_{11} \)
is non-nilpotent. By a direct computation, \([e_4 + 2e_5 - e_8, x, y] = 0\) for all \(x, y \in J_{11}\). Then proof is completed.

**Theorem 2.3** The subalgebra \(H = \langle e_1, e_4, e_5, e_8 \rangle\) is a Cartan subalgebra of the 3-Lie algebra \(J_{11}\). And the decomposition of \(J_{11}\) associate to \(H\) is \(J_{11} = H + J_α + J_−α\), where \(J_α = \langle e_2, e_6 \rangle\), \(J_−α = \langle e_3, e_7 \rangle\), where the linear function \(α : H ∧ H → F\) defined by \(α(1, 4) = 2, α(1, 8) = 2, α(1, 5) = −1\), and others are zero.

**Proof** Define linear function \(α : H ∧ H → F\) by \(α(1, 4) = 2, α(1, 8) = 2, α(1, 5) = −1\), and others are zero. By the multiplication (2) we have \([e_i, e_j, e_l] = α(e_i, e_j)e_2, [e_i, e_j, e_6] = α(e_i, e_j)e_6, [e_i, e_6, e_3] = −α(e_i, e_j)e_3, [e_i, e_j, e_7] = −α(e_i, e_j)e_7\), for all \(e_i, e_j \in H\). Then we have \(J_α = \langle e_2, e_6 \rangle\), \(J_−α = \langle e_3, e_7 \rangle\), and \(J_{11} = H + J_α + J_−α\). The proof is completed.

Now we study the inner derivation algebra \(adJ_{11}\). For \(e_i, e_j ∈ Ω\), denote

\[
ad(e_i, e_j)e_k = \sum_{l=1}^{8} a^{ij}_{kl} e_l, \text{ where } a^{ij}_{kl} = −a^{ji}_{lk} ∈ F.
\]

Then the matrix form of \(ad(e_i, e_j)\) in the basis \(e_1, \ldots, e_8\) is \(\sum_{k,l=1}^{8} a^{ij}_{kl} E_{kl}\), where \(E_{kl}\) are the matrix units.

**Theorem 2.4** Let \(J_{11}\) be a 3-Lie algebra in Theorem 2.2. Then we have

1) \(dim(adJ_{11}) = 12\), and \(X_1 = E_{34} + 2E_{42} + E_{52}, X_2 = E_{24} + 2E_{43} − E_{53}, X_3 = 2E_{22} − 2E_{33}, X_4 = −E_{56} + E_{78} − 2E_{86}, X_5 = E_{57} − E_{68} + 2E_{87}, X_6 = 2E_{66} − 2E_{77}, X_7 = E_{14} + E_8 = E_{12}, X_9 = E_{13}, X_{10} = E_{16}, X_{11} = E_{17}, X_{12} = E_{18}\) is a basis of \(adJ_{11}\). And the multiplication in it is

\[
[X_2, X_1] = X_3, [X_3, X_2] = 2X_2, [X_3, X_1] = −2X_1, [X_6, X_1] = −2X_4,
[X_5, X_4] = X_6, [X_6, X_5] = 2X_5, [X_1, X_7] = 2X_8, [X_1, X_9] = −X_7, [X_2, X_7] = −2X_9, [X_3, X_9] = 2X_9, [X_4, X_{11}] = −X_{12}, [X_4, X_{12}] = 2X_{10}, [X_5, X_{10}] = X_{12},
[X_5, X_{12}] = −2X_{11}, [X_6, X_{10}] = −2X_{10}, [X_6, X_{11}] = 2X_{11}, [X_2, X_8] = X_7,
[X_3, X_8] = −2X_8.
\]

2) \(adJ_{11}\) is a decomposable Lie algebra, and

\[
adJ_{11} = L_1 + L_2, \ [L_1, L_1] = L_1, [L_2, L_2] = L_2, [L_1, L_2] = 0,
\]

where \(L_1 = \langle X_1, X_2, X_3, X_7, X_8, X_9 \rangle\), \(L_2 = \langle X_4, X_5, X_6, X_{10}, X_{11}, X_{12} \rangle\), \(X_1, X_2, X_3 \rangle = \langle X_4, X_5, X_6 \rangle = sl_2\), and \(I_1 = \langle X_7, X_8, X_9 \rangle, I_2 = \langle X_{10}, X_{11}, X_{12} \rangle\) are minimal ideals of \(adJ_{11}\).

**Proof** By a direct computation according to Eq.(2) we have

\[
ad(e_1, e_2) = E_{34} − 2E_{42} + E_{52}, ad(e_1, e_3) = −E_{24} + 2E_{43} − E_{53}, ad(e_1, e_4) = 2E_{22} − 2E_{33}, ad(e_1, e_6) = −E_{56} + E_{78} − 2E_{86}, ad(e_1, e_7) = E_{57} − E_{68} + 2E_{87},
\]

\[
ad(e_1, e_8) = 2E_{66} − 2E_{77}, ad(e_2, e_3) = E_{14}, ad(e_2, e_5) = E_{12}, ad(e_3, e_5) = −E_{13},
ad(e_5, e_6) = E_{16}, ad(e_5, e_7) = −E_{17}, ad(e_6, e_7) = E_{18}.
\]

Then \(\{X_1, \ldots, X_{12}\}\) is a basis of \(adJ_{11}\). From

\[
[ad(e_i, e_j), ad(e_k, e_l)] = ad(e_i, e_j)[e_k, e_l] + ad(e_k, e_l)[e_i, e_j],
\]

we have the result.
At the last of the paper, we discuss the derivation algebra $\text{Der} J_{11}$.

**Theorem 2.5** The derivation algebra $\text{Der} J_{11}$ satisfies:

1) The dimension of $\text{Der} J_{11}$ is 15, and $\text{Der} J_{11}$ with a basis $\{X_1, \cdots, X_{15}\}$, where $X_{13} = E_{11} - 2E_{33} - E_{14} - E_{55} - 2E_{77} - E_{88}$, $X_{14} = E_{54} + 2E_{55} - E_{88}$, $X_{15} = E_{15}$, $X_i$ is in Theorem 2.4 for $1 \leq i \leq 12$. And the multiplication in the basis is

$$
\begin{align*}
[X_2, X_1] &= X_3, [X_{10}, X_{13}] = -X_{10}, [X_5, X_{12}] = -2X_{11}, \\
[X_6, X_5] &= 2X_5, [X_6, X_4] = -2X_4, [X_1, X_7] = 2X_8, [X_1, X_9] = -X_7, \\
[X_2, X_7] &= -2X_9, [X_2, X_8] = X_7, [X_3, X_8] = -2X_8, [X_3, X_9] = 2X_9, \\
[X_4, X_{11}] &= -X_{12}, [X_4, X_{12}] = 2X_{10}, [X_5, X_{10}] = X_{12}, \\
[X_3, X_2] &= 2X_2, [X_6, X_{10}] = -2X_{10}, [X_6, X_{11}] = 2X_{11}, \\
[X_1, X_{13}] &= X_1, [X_2, X_{13}] = -X_2, [X_4, X_{13}] = X_4, [X_5, X_{13}] = -X_5, \\
[X_7, X_{13}] &= -2X_7, [X_8, X_{13}] = -X_8, [X_9, X_{13}] = -3X_9, \\
[X_3, X_1] &= -2X_1, [X_{11}, X_{13}] = -3X_{11}, [X_{12}, X_{13}] = -2X_{12}, \\
[X_2, X_{15}] &= X_9, [X_4, X_{15}] = X_{10}, [X_5, X_{15}] = -X_{11}, [X_{13}, X_{15}] = 2X_{15}, \\
[X_5, X_4] &= X_6, [X_{14}, X_{15}] = -X_7 - 2X_{15} + X_{12}, [X_1, X_{15}] = -X_8.
\end{align*}
$$

2) $\text{Der} J_{11}$ is an indecomposable Lie algebra, and

$$\text{Der} J_{11} = \text{ad} J_{11} + W,$$

where $W = \langle X_{13}, X_{14}, X_{15} \rangle$.

3) Derived algebra $\text{Der}^1 J_{11} = \langle X_1, \cdots, X_{12}, X_{15} \rangle$, $I_1, I_2$ are minimal ideals of $\text{Der} J_{11}$, $L_1, L_2$ are ideals of $\text{Der} J_{11}$ and $[W, L_1] \subseteq L_1, [W, L_2] \subseteq L_2$.

**Proof** For all $D \in \text{Der} J_{11}$, suppose $D(e_i) = \sum_{j=1}^{8} a_{ij} e_j$, $1 \leq i \leq 8$, then the matrix of $D$ in the basis $\{e_1, \cdots, e_8\}$ is $A = (a_{ij})_{i,j=1}^{8} = \sum_{i,j=1}^{8} a_{ij} E_{ij}$, where $E_{ij}$ are $(8 \times 8)$ matrix units, $1 \leq i, j \leq 8$. By s direct computation according to the multiplication (2), we have the result 1).

Thanks to Theorem 2.5, $W = \langle X_{13}, X_{14}, X_{15} \rangle$ are exterior derivations. Then we have $\text{Der} J_{11} = \text{ad} J_{11} + W$.

By a direct computation, $\text{Der}^1 J_{11} = \langle X_2, \cdots, X_{12}, X_{15} \rangle$ and $L_1, L_2$ defined in Theorem 2.5 are ideals of $\text{Der} J_{11}$, and $I_1, I_2$ are minimal ideals.

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**References**


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