Strong Convergence of Mann Iteration for a Hybrid Pair of Mappings in a Banach Space

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Abstract
We prove the strong convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap $f$ and a multi-valued $f$-nonexpansive mapping $T$ in Banach space $E$. Our result extend Theorem 2.3 of Song and Wang [Y. Song, H. Wang, Convergence of iterative algorithms for multi-valued mappings in Banach spaces, Nonlinear Analysis, 70 (2009), 1547–1556] to a hybrid pair of maps.

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1 Introduction

Let $E$ be a Banach space and $K$, a nonempty subset of $E$. We denote by $2^E$, the family of all subsets of $E$; $CB(E)$, the family of nonempty closed and bounded subsets of $E$ and $C(E)$, the family of nonempty compact subsets of $E$. Let $f : K \to K$ be a selfmap. Let $H$ be a Hausdorff metric on $CB(E)$. That is, for $A, B \in CB(E)$,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right\},$$
where
\[ d(x, B) = \inf \{ \| x - y \| : y \in B \}. \]

A multi-valued mapping \( T : K \to 2^K \) is called \( f \)-nonexpansive if
\[ H(Tx, Ty) \leq \| fx - fy \|, \]
for all \( x, y \in K \).

If \( f = I_K \), the identity mapping on \( K \), then we call \( T \) is a multi-valued nonexpansive mapping.

A point \( x \) is a fixed point of \( T \) if \( x \in Tx \). A point \( x \) is called a common fixed point of \( f \) and \( T \) if \( fx = x \in Tx \).

\( F(T) = \{ x \in K : x \in Tx \} \) stands for the fixed point set of a mapping \( T \) and \( F = F(T) \cap F(f) = \{ x \in K : fx = x \in Tx \} \) stands for the common fixed point set of maps \( f \) and \( T \).

Recently, Song and Wang [2] introduced the following Mann iterates of a Multi-valued mapping \( T \):

Let \( K \) be a nonempty convex subset of a Banach space \( E \), \( \alpha_n \in [0,1] \) and \( \gamma_n \in (0, \infty) \) such that \( \lim_{n \to \infty} \gamma_n = 0 \). Let \( T : K \to CB(K) \) be a multi-valued mapping. Let \( x_0 \in K \), and
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n, \quad (1) \]
where \( y_n \in Tx_n \) such that \( \| y_{n+1} - y_n \| \leq H(Tx_{n+1}, Tx_n) + \gamma_n, n = 0, 1, 2, \ldots \).

Song and Wang [2] established the following theorems on the convergence of Mann iteration.

**Theorem 1.1** (Theorem 2.3, Song and Wang [2]). Let \( K \) be a nonempty, compact and convex subset of a Banach space \( E \). Suppose that \( T : K \to CB(K) \) is a multi-valued nonexpansive mappings for which \( F(T) \neq \emptyset \) and for which \( T(y) = \{ y \} \) for each \( y \in F(T) \). For \( x_0 \in K \), let \( \{ x_n \} \) be the Mann iteration defined by (1). Assume that
\[ 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1. \]
Then the sequence \( \{ x_n \} \) strongly converges to a fixed point of \( T \).

The aim of this paper is to prove the strong and weak convergence of Mann iteration for a hybrid pair of maps to a common fixed point of a selfmap \( f \) and a multi-valued \( f \)-nonexpansive mapping \( T \) in Banach space \( E \). Our results extend the results of Song and Wang [2] to a hybrid pair of maps.
2 Preliminary Notes

Throughout this paper $E$ denotes real Banach space. We denote the strong convergence of $\{x_n\}$ to $x$ in $E$ by $x_n \to x$.

**Lemma 2.1** (Nadler [1]). Let $(E, d)$ be a complete metric space, and $A, B \in CB(E)$ and $a \in A$. Then for each positive number $\varepsilon$, there exists $b \in B$ such that
\[
d(a, b) \leq H(A, B) + \varepsilon.
\]

**Lemma 2.2** (Suzuki [3]). Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space $E$ and $\beta_n \in [0, 1]$ with
\[
0 < \lim\inf_{n \to \infty} \beta_n \leq \lim\sup_{n \to \infty} \beta_n < 1.
\]
Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n$ for all integers $n \geq 1$ and
\[
\lim\sup_{n \to \infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

We will construct the following iteration. Let $K$ be a nonempty subset of a metric space $X$. Let $f : K \to K$, $T : K \to CB(K)$ with $f(K)$ is convex and $Tx \subseteq f(K)$ for all $x \in K$. Let $\alpha_n \in [0, 1]$, and $\gamma_n \in (0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$. Choose $x_0 \in K$ and $y_0 \in Tx_0$. Let $z_0 = fx_0$ and
\[
z_1 = fx_1 = (1 - \alpha_0)fx_0 + \alpha_0 y_0
\]
\[
= (1 - \alpha_0)z_0 + \alpha_0 y_0.
\]
From Lemma 2.1, there exists $y_1 \in Tx_1$ such that
\[
\|y_1 - y_0\| \leq H(Tx_1, Tx_0) + \gamma_0.
\]
Let
\[
z_2 = fx_2 = (1 - \alpha_1)z_1 + \alpha_1 y_1.
\]
Inductively, we have
\[
z_{n+1} = fx_{n+1} = (1 - \alpha_n)z_n + \alpha_n y_n,
\]
(2)
where $y_n \in Tx_n$ such that
\[
\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n, \quad n = 0, 1, 2, \ldots.
\]
3 Main Results

These are the main results of the paper.

Proposition 3.1 Let $K$ be a nonempty subset of a Banach space $E$. Let $f : K \to K$ be a selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multi-valued $f$-nonexpansive mapping and $Tx \subseteq f(K)$ for all $x \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps $T$ and $f$, defined by (2) and assume also that

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.$$ 

Then $\lim_{n \to \infty} \|z_n - y_n\| = 0$ and $\lim_{n \to \infty} d(z_n, Tx_n) = 0$.

Proof. From the definition of the Mann iteration $\{z_n\}$ given by (2), it follows that $z_{n+1} = (1 - \alpha_n)z_n + \alpha_n y_n$, where $y_n \in Tx_n$ such that

$$\|y_{n+1} - y_n\| \leq H(Tx_{n+1}, Tx_n) + \gamma_n$$

$$\leq \|z_{n+1} - z_n\| + \gamma_n, \quad n = 0, 1, 2, \ldots.$$ 

Therefore,

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|z_{n+1} - z_n\|) \leq \limsup_{n \to \infty} \gamma_n = 0.$$ 

Hence, all conditions of Lemma 2.2 are satisfied. Hence, by Lemma 2.2, we obtain $\lim_{n \to \infty} \|z_n - y_n\| = 0$.

Since $y_n \in Tx_n$ for all $n = 0, 1, 2, \ldots$, we have $d(z_n, Tx_n) \leq \|z_n - y_n\|$.

Hence, $\lim_{n \to \infty} d(z_n, Tx_n) = 0$.

Theorem 3.2 Let $K$ be a nonempty compact subset of a Banach space $E$. Let $f : K \to K$ be a continuous selfmap with $f(K)$ is convex. Suppose $T : K \to CB(K)$ is a multi-valued $f$-nonexpansive mapping for which $Tx \subseteq f(K)$ for all $x \in K$; $F(T) \cap F(f) \neq \emptyset$, and $d(x, Tx) \leq d(fx, Tx)$ for all $x, y \in K$. For $x_0 \in K$, let $\{z_n\}$ be the Mann iteration associated with the maps $T$ and $f$, defined by (2) and assume also that

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.$$ 

If $T(y) = \{y\}$ for each $y \in F(T)$, then the Mann iteration $\{z_n\}$ strongly converges to a common fixed point of $f$ and $T$. 

Proof. It follows from Proposition 3.1 that \( \lim_{n \to \infty} d(z_n, Tx_n) = 0 \).
Further, since \( d(x_n, Tx_n) \leq d(z_n, Tx_n) \) we get \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \).
Now let \( p \in F(T) \cap F(f) \). Then,
\[
\|z_{n+1} - p\| \leq (1 - \alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\|
\leq (1 - \alpha_n)\|z_n - p\| + \alpha_nH(Tx_n, Tp)
\leq (1 - \alpha_n)\|z_n - p\| + \alpha_n\|z_n - p\|
= \|z_n - p\|, \ n = 0, 1, 2, \ldots.
\]
Then the sequence \( \{\|z_n - p\|\} \) is a decreasing sequence of nonnegative reals and
hence \( \lim_{n \to \infty} \|z_n - p\| \) exists for each \( p \in F(T) \cap F(f) \).

From the compactness of \( K \), there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( \lim_{k \to \infty} x_{n_k} = u \) for some \( u \in K \). By the continuity of \( f \), we have
\[
\lim_{k \to \infty} z_{n_k} = fu = q \text{ (say).}
\]
Now
\[
d(q, Tu) \leq \|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tu)
\leq \|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) + \|fu - fx_{n_k}\|
= 2\|q - z_{n_k}\| + d(z_{n_k}, Tx_{n_k}) \to 0 \text{ as } k \to \infty.
\]
Hence, \( fu = q \in Tu \).

Also,
\[
d(u, Tu) \leq \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + H(Tx_{n_k}, Tu)
\leq \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \|fu - fx_{n_k}\|
= \|u - x_{n_k}\| + d(x_{n_k}, Tx_{n_k}) + \|z_{n_k} - q\| \to 0 \text{ as } k \to \infty.
\]
Hence, \( u \in Tu \) so that \( Tu = \{u\} \).
Hence, \( fq = q \in Tq \).

Thus, \( q \) is a common fixed point of \( f \) and \( T \).
Now replacing \( q \) in place of \( p \), we get that \( \lim_{n \to \infty} \|z_n - q\| \) exists and hence
\[
\lim_{n \to \infty} \|z_n - q\| = 0.
\]
Hence the conclusion follows.

Corollary 3.3 If \( f = I_K \), the identity mapping on \( K \), we get Theorem 1.1.
Hence, Theorem 3.2 extends Theorem 1.1 to a hybrid pair of maps.

The following is an example in support of Theorem 3.2.
Example 3.4 Let $E = R$, the set of all real numbers, with the usual norm and $K = [\frac{1}{3}, 1]$. We define mappings $f : K \rightarrow K$ by $fx = 1 - \frac{1}{2}x$ and $T : K \rightarrow CB(K)$ by $Tx = [\frac{2}{3}, \frac{5}{6}x + \frac{1}{4}]$.

Here $f(K) = [\frac{1}{2}, \frac{2}{3}]$, $Tx \subseteq f(K)$ for all $x \in K$, and $F(f) \cap F(T) = \{\frac{2}{3}\} \neq \emptyset$.

Now we consider the following two cases.

Case (i): $x \in [\frac{2}{3}, 1]$.

Then, $fx = 1 - \frac{1}{2}x \geq \frac{2}{3}$, $Tx = [\frac{1}{2}x + \frac{1}{4}, \frac{2}{3}]$.

Thus we have $d(x, Tx) = \frac{1}{2}(\frac{2}{3} - x) = d(fx, Tx)$.

Case (ii): $x \in \left[\frac{2}{3}, \frac{1}{2}\right]$.

Then, $fx = 1 - \frac{1}{2}x \leq \frac{2}{3}$, $Tx = [\frac{2}{3}, \frac{1}{2}x + \frac{1}{4}]$.

Thus we have $d(x, Tx) = \frac{1}{2}(x - \frac{2}{3}) = d(fx, Tx)$.

Hence, from case (i) and case (ii), it follows that

$$d(x, Tx) = d(fx, Tx) \text{ for all } x \in K.$$ 

Also, $T$ is $f$-nonexpansive on $K$, for, proceeding as in the above, we get

$$H(Tx, Ty) = \max\{\sup_{a \in Ty} d(Tx, a), \sup_{a \in Tx} d(a, Ty)\}$$

$$= |fx - fy| \text{ for all } x, y \in K;$$

and $Ty = \{y\}$ for each $y \in F(T) = \{\frac{2}{3}\}$.

Next we show that for any $x_0 \in K$, the Mann iteration defined by (2) converges to the unique common fixed point of $f$ and $T$, which is the conclusion of Theorem 3.2.

Let $x_0 \in K$ be arbitrary. Let $\alpha_n \in [0, 1]$ be such that

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1.$$ 

If $x_0 \in [\frac{1}{3}, \frac{2}{3}]$. Then $fx_0 = 1 - \frac{1}{2}x_0$ and $Tx_0 = [\frac{1}{3}x_0 + \frac{1}{4}, \frac{2}{3}]$. Choose $y_0 = \frac{1}{2}x_0 + \frac{1}{4}$. Then $y_0 \in TTx_0$, and $fx_1 = \frac{2}{3} + (\frac{1}{2} - \alpha_0)\left(\frac{2}{3} - x_0\right)$.

On continuing this process, inductively we get a sequence $\{x_n\}$ in $K$ such that

$$fx_{n+1} = \frac{2}{3} + \frac{1}{2}\left(\frac{2}{3} - x_0\right) \prod_{j=0}^{n}(1 - 2\alpha_j), \; n = 0, 1, 2, \ldots.$$ (3)

If $x_0 \in [\frac{2}{3}, 1]$. Then $fx_0 = 1 - \frac{1}{2}x_0$ and $Tx_0 = [\frac{2}{3}, \frac{1}{2}x_0 + \frac{1}{4}]$. Again, choose $y_0 = \frac{1}{2}x_0 + \frac{1}{4}$. Then $y_0 \in TTx_0$, and $fx_1 = \frac{2}{3} - (\frac{1}{2} - \alpha_0)\left(x_0 - \frac{2}{3}\right)$.

On continuing this process, inductively we get a sequence $\{x_n\}$ in $K$ such that

$$fx_{n+1} = \frac{2}{3} - \frac{1}{2}(x_0 - \frac{2}{3}) \prod_{j=0}^{n}(1 - 2\alpha_j), \; n = 0, 1, 2, \ldots.$$ (4)
Since $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$, there exist real numbers $0 < \gamma, \eta < 1$ such that $0 < \gamma \leq \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n \leq \eta < 1$, and hence there exists a positive integer $N$ such that $\gamma \leq \alpha_n \leq \eta$ for all $n \geq N$.

Hence, $\beta = \sup_{j \geq N} |2\alpha_j - 1| \leq \max\{|2\gamma - 1|, |2\eta - 1|\} < 1$.

Now, by using (3) and (4) for $x_0 \in K$, we get

$$fx_{n+1} = \frac{2}{3} + \frac{1}{2} \left(\frac{2}{3} - x_0\right) \prod_{j=0}^{N-1} (1 - 2\alpha_j) \prod_{j=N}^{n} (1 - 2\alpha_j), \quad n \geq N. \quad (5)$$

Hence,

$$|fx_{n+1} - \frac{2}{3}| \leq \frac{1}{2} \left(\frac{2}{3} - x_0\right) \prod_{j=0}^{N-1} |1 - 2\alpha_j| \prod_{j=N}^{n} |1 - 2\alpha_j| \leq \frac{1}{2} \left(\frac{2}{3} - x_0\right) \prod_{j=0}^{N-1} |1 - 2\alpha_j| \beta^{n-N+1}, \quad n \geq N. \quad (6)$$

Hence, $fx_n \to \frac{2}{3}$ strongly as $n \to \infty$, and $\frac{2}{3}$ is a common fixed point of $f$ and $T$.

**References**


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