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Some Results on $T^i - \Gamma - AG$ $(i = 1, 2, 3, 4)$ groupoids

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Abstract

Non-associatione algebraic stractures are of interest to consider for their remarkable properties. In this paper, we generalize notions of the $T^i - AG$-groupoids to $T^i - \Gamma - AG$-groupoids. Then we investigate some properties of $T^i - \Gamma - AG$-groupoids $(i = 1, 2, 3, 4)$ and prove that every $T^1 - \Gamma - AG$-groupoid is $\Gamma -$paramedical, every $T^2 - \Gamma - AG$- groupoid is transitively commutative, every $\Gamma - AG$- band is $T^3 - \Gamma - AG$- groupoid and every $T^4 - \Gamma - AG$- groupoid with a left identity is a $Bol^* - \Gamma - AG$-groupoid.

Keywords: $\Gamma -$semigroup, $\Gamma - AG$-groupoid, $T^i - \Gamma - AG$-groupoids $(i = 1, 2, 3, 4)$, nuclear square and $Bol^* - \Gamma - AG$-groupoid.

1.Introduction

The idea of generalization of communicative semigroups was introduced in 1977 by M.A.Kazim and M.Naseerudin. They named this structure as the left almost semigroup (LA-semigroup) in [2]. It is also called as Abel-Grassmann'sgroupoid (AG-groupoid) in [1,2]. In generalizing this notion the new structure $\Gamma - AG$-groupoid is also defined by T.Shah and Rahman in [6]. In this paper we extend certain properties of AG-groupoid to $\Gamma - AG$-groupoid.
Some new results on T^1, T^2 and $T^4 - AG$- groupoids have been recently studied by Ahmad [3]. We generalize these results and investigate some properties of T^1,

T^2 and $T^4-\Gamma-AG$ - groupoids, and also study the T^3 property.
Let S and Γ be non-empty sets we call S to be a $\Gamma-$semigroup if there exists a mapping $S\times\Gamma\times S\to S$ writing (a,γ,b) by $a\gamma\mathrm{b}$, such that S satisfies the identity $(a\gamma\mathrm{b})\beta c=a\gamma(b\beta c)$ for all $a,\mathrm{b},\mathrm{c}\in\mathrm{S}$ and $\gamma,\beta\in\Gamma$.
Following [5,6]we first recall the preliminary definitions:
Definition 1.1.[6] Let S and Γ be non-empty sets we call S to be a $\Gamma-AG$-groupoid if there exists a mapping $S\times\Gamma\times S\to S$ writing (a,γ,b) by $a\gamma\mathrm{b}$, such that S satisfies the identity $(a\gamma\mathrm{b})\beta c=(c\gamma\mathrm{b})\beta a$ for all $a,\mathrm{b},\mathrm{c}\in\mathrm{S}$ and $\gamma,\beta\in\Gamma$.
Definition 1.2.[6] An element $e\in S$is called a left identity of Γ -AG-groupoid if $e\gamma a=a$ for all $a\in$Sandγ∈Γ.
Definition 1.3.[5]A $\Gamma-AG$-groupoids is called Γ-medial if for every $a,\mathrm{b},\mathrm{c},d\in\mathrm{S}$ and $\gamma,\beta\in\Gamma$, $(a\alpha\mathrm{b})\beta(c\gamma d)=(a\alpha c)\beta(b\gamma d)$.
Definition 1.4.[5] A $\Gamma-AG$-groupoids is called Γ-Paramedial if for every $a,\mathrm{b},\mathrm{c},d\in\mathrm{S}$ and $\gamma,\beta\in\Gamma$, $(a\alpha\mathrm{b})\beta(c\gamma d)=(d\alpha\mathrm{b})\beta(c\gamma a)$.
Definition 1.5.[5] A $\Gamma-AG$-groupoid S is called a locally associative if for every $a\in\mathrm{S}$and $\beta,\gamma\in\Gamma$ it satisfies $(a\gamma a)\beta a=a\gamma(a\beta a)$.
Definition 1.6.[5] An element a of $\Gamma-AG$-groupoid S is called$\{\gamma\}$-idempotent that $\gamma\in\Gamma$ if $a\gamma a=a$.
Definition 1.7.[5] A $\Gamma-AG$-groupoid S is called a Γ- idempotent if every their element be $\{\gamma\}$-idempotent for every $\gamma\in\Gamma$.
In the following we introduce certain definitions which are in fact the generalizations of the definitions of the references[4-8].
Definition 1.8. A $\Gamma-AG$-groupoid is called a $T^1-\Gamma-AG$-groupoid if for every $a,\mathrm{b},\mathrm{c},\mathrm{d}\in\mathrm{S}$, $\gamma\in\Gamma$, $a\gamma b=c\gamma d$ implies $b\gamma a=d\gamma c$.
Definition 1.9. A $\Gamma-AG$-groupoid is called a $T^2-\Gamma-AG$-groupoid if for every $a,\mathrm{b},\mathrm{c},\mathrm{d}\in\mathrm{S}$, $\gamma\in\Gamma$, $a\gamma b=c\gamma d$ implies $a\gamma c=b\gamma d$.
Definition 1.10. A $\Gamma-AG$-groupoid S, for every $a,\mathrm{b},\mathrm{c},\mathrm{d}\in\mathrm{S}$, $\gamma\in\Gamma$ is called a
(i) $T_l^3-\Gamma-AG$-groupoid, if $a\gamma b=a\gamma c$ implies $b\gamma a=c\gamma a$,
(ii) $T_r^3-\Gamma-AG$-groupoid, if $b\gamma a=c\gamma a$ implies $a\gamma b=a\gamma c$,
(iii) $T^3-\Gamma-AG$-groupoid, if it is both T_l^3 and $T_r^3-\Gamma-AG$-groupoids.
Definition 1. 11. A $\Gamma-AG$-groupoid for every $a,\mathrm{b},\mathrm{c},\mathrm{d}\in\mathrm{S}$, $\gamma\in\Gamma$ is called a
(i) $T_f^4-\Gamma-AG$-groupoid, if $a\gamma b=c\gamma d$ implies $a\gamma d=c\gamma b$,
(ii)$T_b^4-\Gamma-AG$-groupoid, if $a\gamma b=c\gamma d$ implies $d\gamma a=b\gamma c$,
(iii) $T^4-\Gamma-AG$-groupoid, if it is both T_f^4 and $T_b^4-\Gamma-AG$-groupoid.
Definition 1.12. A $\Gamma-AG$-groupoid for every $a,\mathrm{b},\mathrm{c},\mathrm{d}\in\mathrm{S}$, $\alpha,\beta,\gamma\in\Gamma$ is called a

(i) Left nuclear square, if $a^2_\alpha\beta(\mathrm{b}\gamma\mathrm{c})=(a^2_\alpha\beta\mathrm{b})\gamma\mathrm{c}$, that $a^2_\alpha=a\alpha a$,

(ii)Right nuclear square, if $(a\alpha b)\gamma c^2_\beta=a\alpha(b\gamma c^2_\beta)$,

(iii) Middle nuclear square, if $(a\alpha b^2_\beta)\gamma c=a\alpha(b^2_\beta\gamma c)$,

(iv) Nuclear square, if it is left, right and middle nuclear square.

We recall the three following lemmas from [4,5] which are applied to get some results.

Lemma 1. 1. Every $\Gamma-AG$-groupoid is Γ-medial.

Lemma 1.2. Every $\Gamma-AG$-groupoid with left identity is Γ-paramedial.

Lemma 1.3. In an $\Gamma-AG$-groupoid S with left identity, we have $a\alpha(\mathrm{b}\beta c)=b\alpha(a\beta c)$ for every $a,b,c\in\mathrm{S}$ and $\alpha,\beta\in\Gamma$.

Lemma 1.4. In a $\Gamma-AG$-groupoid S with a left identity, we have $a\alpha b=a\beta b$ for every $a,b\in\mathrm{S}$ and $\alpha,\beta\in\Gamma$.

Proof. Let S be a $\Gamma-AG$-groupoid with a left identity e, also for all $a,b\in\mathrm{S}$ and $\alpha,\beta\in\Gamma$

$a\alpha b=a\alpha(\mathrm{e}\beta\mathrm{b})$ (by left identity)

$=e\alpha(a\beta\mathrm{b})$ (by lemma (1.3))

$=a\beta\mathrm{b}$ (by left identity)

Lemma 1. 5. Let S be $\Gamma-AG$-groupoid then $(a\beta\mathrm{b})^2_\gamma=a^2_\beta\gamma b^2_\beta$ for every $a,b\in\mathrm{S}$ and $\beta,\gamma\in\Gamma$.

Proof. Let S be $\Gamma-AG$-groupoid, and for every $a,b\in\mathrm{S}$ and $\beta,\gamma\in\Gamma$

$(a\beta\mathrm{b})^2_\gamma=(a\beta\mathrm{b})\gamma(a\beta\mathrm{b})$

$=(a\beta a)\gamma(b\beta\mathrm{b})$ (by $\Gamma-$ medial law)

$=a^2_\beta\gamma b^2_\beta$.

2. Properties of $T^i-\Gamma-AG$ -groupoids($i=1,2,3,4$)

In this section, we generalize notions of the T^i-AG-groupoids to $T^i-\Gamma-AG$-groupoids.

Then we investigate some properties of $T^i-\Gamma-AG$-groupoids($i=1,2,3,4$).

Proposition 2.1. Every $\Gamma-$paramedial S with a left identity e is a left nuclear square $\Gamma-AG$-groupoid.

Proof. Let $a,b,c\in\mathrm{S}$ and $\alpha,\beta,\gamma\in\Gamma$ .Then

$a^2_\alpha\beta(\mathrm{b}\gamma\mathrm{c})=(a\alpha a)\beta(b\gamma c)=(a\alpha b)\beta(a\gamma c)$ (by lemma1. 1.)

$=(c\alpha a)\beta(b\gamma a)$ (by $\Gamma-$ paramedial law)

$=(c\alpha b)\beta(a\gamma a)$ (by lemma1. 1.)

$=((a\gamma a)\alpha\mathrm{b})\beta c$ (by left invertive)

$=((a\alpha a)\beta\mathrm{b})\gamma c$ (by lemma1. 4.)

$= (a_{\alpha}^{2}\beta \text{b})\gamma \text{c}$.

Hence, S is a left nuclear square $\Gamma - AG$ -groupoid.

Proposition 2.2. Every $T^1 - \Gamma - AG$ -groupoid is $\Gamma -$ paramedial, but not vice-versa.

Proof. Let S be a $T^1 - \Gamma - AG$ -groupoid and let $a,b,c,d \in \text{S}$ and $\alpha, \beta, \gamma \in \Gamma$. Now we have

$(a\alpha b)\gamma(\text{c}\,\beta d) = (a\alpha c)\gamma(\text{b}\,\beta d)$ (by lemma1. 1.)

$\Rightarrow (\text{c}\,\beta d)\gamma(a\alpha b) = (\text{b}\,\beta d)\gamma(a\alpha c)$ (by $T^1 - \Gamma - AG$ -groupoid)

$(\text{c}\,\beta d)\gamma(a\alpha b) = (\text{b}\,\beta a)\gamma(d\alpha c)$ (by lemma1. 1.)

$\Rightarrow (a\alpha b)\gamma(\text{c}\,\beta d) = (d\alpha c)\gamma(\text{b}\,\beta a)$ (by $T^1 - \Gamma - AG$ -groupoid)

$(a\alpha b)\gamma(\text{c}\,\beta d) = (d\alpha b)\gamma(c\beta a)$.(by lemma1. 1.)

Hence, S is $\Gamma -$ paramedial.The following example shows that the converse is not valid:

Consider $S = \{a,b,c\}$ and $\Gamma = \{\gamma\}$ with the following table.Then S is $\Gamma -$ paramedical,but we have $b = c\gamma a = c\gamma c, a = a\gamma c \neq c\gamma c = b$ or $b\gamma c = a\gamma b, c\gamma b \neq b\gamma a$ i.e .S is not a $T^1 - \Gamma - AG$ -groupoid.

γ	a	b	c
a	a	a	a
b	a	a	a
c	b	b	b

Corollary 2.1. Every $T^1 - \Gamma - AG$ -groupoid with a left identity is a left nuclear square $\Gamma - AG$ -groupoid.

Proof. By propositions2.1and2.2, the result is immediate.

Definition 2.1. A $\Gamma - AG$ -groupoid S is called a $Bol^* - \Gamma - AG$ -groupoid if it satisfies the identity $a\alpha((\text{b}\,\beta\,\text{c})\gamma\,\text{d}) = ((a\alpha\,\text{b})\beta\,\text{c})\gamma\,\text{d}$, for all $a,\text{b},\text{c},\text{d} \in \text{S}$ and $\alpha, \beta, \gamma \in \Gamma$.

Proposition 2.3 Every $Bol^* - \Gamma - AG$ -groupoid with a left identity is a $T^1 - \Gamma - AG$ -groupoid.

Proof. Aame that S be a $Bol^* - \Gamma - AG$ -groupoid with left identity e for every $a,b,c,d \in \text{S}$ and $\alpha, \beta, \gamma \in \Gamma$. Let $a\alpha b = c\alpha d$ .Then,

$b\alpha a = e\beta((\text{e}\,\gamma\,\text{b})\alpha\,\text{a})$ (by left identity law)

$= ((e\beta\,\text{e})\gamma\,\text{b})\alpha\,\text{a}$ (by $Bol^* - \Gamma - AG$ -groupoid)

$= (e\gamma \text{b})\alpha \text{a}$ (by left identity)

$= (a\gamma \text{b})\alpha e$ (by left invertive law)

$= (a\alpha b)\alpha e$ (by lemma 1.4)

$= (c\alpha d)\alpha e$ (by assumption)

$= (e\alpha d)\alpha c$ (by left invertive)

$= d\alpha c$ (by left identity)

Hence, S is a $T^1 - \Gamma - AG$ -groupoid.

Corollary 2. 2. Every $Bol^* - \Gamma - AG$ -groupoid with left identity is $\Gamma -$ paramedial.

Proof. By Propositions2.2and 2.3, the result is immediate.

Definition 2.2.A $\Gamma - AG$ -groupoid S is called a $\Gamma - AG - 3$ -band if $a\gamma(a\beta a) = (a\gamma a)\beta a = a$, for all $a \in \text{S}$ and $\beta, \gamma \in \Gamma$.

Theorem 2.1. Every $T^1 - \Gamma - AG$ -3-band with left identity is $\Gamma -$ semigroup.

Proof. Let S be a $T^1 - \Gamma - AG$ -groupoid and $\Gamma - AG$ -3-band.For every $a, b, c, d \in \text{S}$ and $\alpha, \beta, \gamma, \delta \in \Gamma$, we get:

$(a\alpha \text{b})\gamma c = (\text{c}\,\alpha\, \text{b})\gamma a$ (by left invertive law)

$c\gamma(a\alpha \text{b}) = a\gamma(\text{c}\,\alpha\, \text{b})$ (by $T^1 - \Gamma - AG$ -groupoid)

$= ((a\beta a)\delta a)\gamma(\text{c}\,\alpha b)$ (by $\Gamma - AG$ -3-band)

$= ((a\beta a)\delta c)\gamma(a\alpha b)$ (by Γ -medial law)

$(a\alpha \text{b})\gamma c = (a\alpha \text{b})\gamma((a\beta a)\delta \text{c})$ (by $T^1 - \Gamma - AG$ -groupoid)

$= (a\alpha(a\beta a))\gamma(b\delta c)$ (by Γ -medial law)

$= a\gamma(\text{b}\,\delta\, \text{c})$ (by $\Gamma - AG$ -3-band)

$= a\alpha(\text{b}\,\gamma\, \text{c})$ (by lemma 1. 4.)

Hence, S is a Γ -semigroup.

Corollary2. 3. Every $Bol^* - \Gamma - AG$ -groupoid with left identity is Left nuclear square $\Gamma - AG$ -groupoid.

Proof. By propositions 2.3 and corollary 2.1, the result is immediate.

Definition 2.3.A $\Gamma - AG$ -groupoid is called transitively commutative if for all $a, \text{b}, \text{c} \in \text{S}$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a,\ b\gamma c = c\gamma b$ imply $a\gamma c = c\gamma a$.

Proposition 2.4. Every $T^2 - \Gamma - AG$ - groupoid is transitively commutative $\Gamma - AG$ - groupoid.

Proof. Let S be a $T^2 - \Gamma - AG$ -groupoid. Then $\forall a, b, c, d \in \text{S}$ and $\gamma \in \Gamma$ suppose $a\gamma b = c\gamma d \Rightarrow a\gamma c = b\gamma d$. Let $a\gamma b = b\gamma a,\ b\gamma c = c\gamma b$.

$a\gamma c = \text{b}\,\gamma d$ (by assumption) (1)

$a\gamma b = c\gamma d$ (by $T^2 - \Gamma - AG$ - groupoid)

$b\gamma a = c\gamma d$ (by assumption)

$b\gamma c = a\gamma d$ (by $T^2 - \Gamma - AG$ - groupoid)
$c\gamma b = a\gamma d$ (by assumption)
$c\gamma a = b\gamma d$ (by $T^2 - \Gamma - AG$ - groupoid) (2)
$\Rightarrow a\gamma c = c\gamma a$ (by equation (1) , (2))
Hence S is transitively commutative $\Gamma - AG$ -groupoid.

Proposition 2.5. Let S be a $\Gamma - AG$ -groupoid with left identity e such that for every $a \in$ S, $\alpha \in \Gamma$, $a\alpha a = a_\alpha^2 = e$. Then S is a $T^2 - \Gamma - AG$ - groupoid.

Proof. Let $a, b, c, d \in$ S and $\alpha, \beta, \gamma \in \Gamma$,such that $a\gamma b = c\gamma d$ (1),
Then we have

$a\gamma c = (e\beta a)\gamma c = (c\beta a)\gamma e$ (by left invertive law)
$= (c\beta a)\gamma(b\alpha b)$ (by assumption)
$= (c\beta b)\gamma(a\alpha b)$ (by Γ -medial law)
$= (c\beta b)\gamma(c\alpha d)$ (by equation (1) and lemma 1. 4.)
$= (c\beta c)\gamma(b\alpha d)$ (by Γ -medial law)
$= e\gamma(b\alpha d)$ (by assumption)
$= b\gamma(e\alpha d) = b\gamma d$ (by left identity)

Hence, S is a $T^2 - \Gamma - AG$ - groupoid.

Proposition 2.6. Every $T^2 - \Gamma - AG$ - groupoid is a $T^1 - \Gamma - AG$ - groupoid.

Proof. Let S be a $T^2 - \Gamma - AG$ -groupoid. Consider for every $a, b, c, d \in$ S and $\gamma \in \Gamma$,

$a\gamma b = c\gamma d$ (by assumption)
$a\gamma c = b\gamma d$ (by $T^2 - \Gamma - AG$ - groupoid)
$b\gamma d = a\gamma c$
$b\gamma a = d\gamma c$ (by $T^2 - \Gamma - AG$ - groupoid)

Hence, S is a $T^1 - \Gamma - AG$ - groupoid.

Theorem 2.2. Every $T^2 - \Gamma - AG$ - groupoid with a left identity is a $Bol^* - \Gamma - AG$ -groupoid.

Proof. Let S be a $T^2 - \Gamma - AG$ -groupoid. Then for every $a, b, c, d \in$ S and $\alpha, \beta, \gamma \in \Gamma$, we have $a\gamma b = c\gamma d$ implies $a\gamma c = b\gamma d$. Now consider

$((a\alpha b)\gamma c)\beta d = (d\gamma c)\beta(a\alpha b)$ (by left invertive law) (1)
$d\beta((a\alpha b)\gamma c) = (a\alpha b)\beta(d\gamma c)$ (by Proposition 2.6)
$d\beta((a\alpha b)\gamma c) = (d\gamma c)\alpha b)\beta a$ (by left invertive law)
$((d\gamma c)\alpha b)\beta a = d\beta((a\alpha b)\gamma c)$ (by $T^2 - \Gamma - AG$ - groupoid)
$((d\gamma c)\alpha b)\beta d = a\beta((a\alpha b)\gamma c)$ (by Proposition 2.6)
$(d\beta(d\gamma c)\alpha b) = (((a\alpha b)\gamma c)\beta a)$ (by $T^2 - \Gamma - AG$ - groupoid)
$a\beta((d\gamma c)\alpha b) = ((a\alpha b)\gamma c)\beta d$ (by theorem 2.1)

$a\beta((d\gamma c)\alpha b) = (d\gamma c)\beta(a\alpha b)$ (by left invertive law) (2)

$((a\alpha b)\gamma c)\beta d = a\beta((d\gamma c)\alpha b)$ (by equal (1), (2))

$((a\alpha b)\gamma c)\beta d = a\beta((b\gamma c)\alpha d)$ (by left invertive law)

$((a\alpha b)\gamma c)\beta d = a\alpha((b\gamma c)\beta d)$ (by lemma 1.4.)

Hence, S is a $Bol^{*} - \Gamma - AG$ - groupoid.

Corollary 2. 4. Every $T^{2} - \Gamma - AG$ - groupoid is Γ -paramedial.

Proof. By Propositions 2.2 and 2.6 , the result is immediate.

Corollary 2.5. Every $T^{2} - \Gamma - AG$ - groupoid with left identity is left nuclear square $\Gamma - AG$ - groupoid.

Proof. By Proposition 2.6 and corollary 2. 1 is obviously.

Definition 2.4. A $\Gamma - AG$ -groupoid is called $\Gamma - AG$ - band if every their elements be $\Gamma -$ idempotent.

Theorem 2.3. Every $\Gamma - AG$ - band is a $T^{3} - \Gamma - AG$ - groupoid.

Proof. Let $a\gamma b = a\gamma c$,for $a,b \in$ S and $\gamma \in \Gamma$,

$b\gamma a = (b\gamma b)\gamma a$ (by Γ -idempotent)

$= (a\gamma b)\gamma b$ (by left invertive law)

$= (a\gamma c)\gamma b$ (by assumption)

$= (a\gamma c)\gamma(b\gamma b)$ (by Γ -idempotent)

$= (a\gamma b)\gamma(c\gamma b)$ (by Γ -medial law)

$= (a\gamma c)\gamma(c\gamma b)$ (by assumption)

$= ((a\gamma a)\gamma c)\gamma(c\gamma b)$ (by Γ - idempotent)

$= ((c\gamma a)\gamma a)\gamma(c\gamma b)$ (by Γ - invertive law)

$= ((c\gamma b)\gamma a)\gamma(c\gamma a)$ (by Γ - invertive law)

$= ((a\gamma b)\gamma c)\gamma(c\gamma a)$ (by Γ - invertive law)

$= ((a\gamma c)\gamma c)\gamma(c\gamma a)$ (by assumption)

$= ((c\gamma c)\gamma a)\gamma(c\gamma a)$ (by Γ - invertive law)

$= (c\gamma a)\gamma(c\gamma a) = c\gamma a$.(by Γ - idempotent)

Hence, S is a $T_{l}^{3} - \Gamma - AG$ - groupoid.

Let $b\beta a = c\beta a$,for every $a,b \in$ S and $\beta \in \Gamma$,

$a\beta b = (a\beta a)\beta b$ (by Γ -idempotent)

$= (b\beta a)\beta a$ (by Γ - invertive law)

$= (c\beta a)\beta a$ (by assumption)

$= (a\beta a)\beta c$ (by Γ - invertive law)

$= a\beta c$.(by Γ -idempotent)

Hence, S is a $T_r^3-\Gamma-AG$ - groupoid. Then S is a $T^3-\Gamma-AG$ - groupoid.

Proposition 2.7. Every $T^1-\Gamma-AG$ - groupoid is a $T^3-\Gamma-AG$ - groupoid.
Proof. Uing their definitions is obviously.

Corollary 2.6. Every $T^2-\Gamma-AG$ - groupoid is $T^3-\Gamma-AG$ - groupoid.
Proof. By Proposition 2.6 and 2.7 is obviously.

Proposition 2.8. Every $T^4-\Gamma-AG$ - groupoid is a transitively commutative $\Gamma-AG$ - groupoid.
Proof. Let $a,b,c,d\in S$ and $\alpha\in\Gamma$ with $a\alpha b=b\alpha a$ and $b\alpha c=c\alpha b$.
Now we have:
$a\alpha a=b\alpha b$ and $b\alpha b=c\alpha c$ (by $T^4-\Gamma-AG$ - groupoid)
$\Rightarrow a\alpha a=c\alpha c$
$a\alpha c=c\alpha a$. (by $T^4-\Gamma-AG$ - groupoid)
Hence, S is a transitively commutative $\Gamma-AG$ -groupoid.

Theorem 2.4. Every $T^4-\Gamma-AG$ - groupoid with left identity is $Bol^*-\Gamma-AG$ - groupoid.
Proof. Let $a,b,c,d\in S$ and S is $T^4-\Gamma-AG$ - groupoid and $e\in S$ and $\alpha,\beta,\gamma\in\Gamma$
$((a\alpha b)\beta c)\gamma d=(d\beta c)\gamma(a\alpha b)$ (by left invertive law)
$((a\alpha b)\beta c)\gamma(a\alpha b)=(d\beta c)\gamma d$ (by $T_f^4-\Gamma-AG$ - groupoid)
$d\gamma((a\alpha b)\beta c)=(a\alpha b)\gamma(d\beta c)$ (by $T_b^4-\Gamma-AG$ - groupoid)
$d\gamma((a\alpha b)\beta c)=((d\beta c)\alpha b)\gamma a$ (by left invertive law)
$d\gamma a=((d\beta c)\alpha b)\gamma((a\alpha b)\beta c)$ (by $T_f^4-\Gamma-AG$ - groupoid)
$((a\alpha b)\beta c)\gamma d=\alpha\gamma((d\beta c)\alpha b)$ (by $T_b^4-\Gamma-AG$ -groupoid)
$((a\alpha b)\beta c)\gamma d=\alpha\gamma((b\beta c)\alpha d)$ (by left invertive law)
$((a\alpha b)\beta c)\gamma d=\alpha\alpha((b\beta c)\gamma d)$ (by lemma 1.4.)
Hence, S is a $Bol^*-\Gamma-AG$ - groupoid.

Theorem 2.5. If a $\Gamma-AG$ - band S contains a left identity e, then S become a commutative Γ -monoid.
Proof. By lemma 2 and remark 3. We have for every $a,b\in S$ and $\gamma\in\Gamma$. Then
$a\gamma b=(e\gamma a)\gamma b$
$=(b\gamma a)\gamma e$ (by left invertive law)
$=((b\gamma a)\gamma(b\gamma a))\gamma e$ (by $\Gamma-$ idempotent)
$=(e\gamma(b\gamma a))\gamma(b\gamma a)$ (by left invertive law)
$=(b\gamma a)\gamma(b\gamma a)=b\gamma a$ (by $\Gamma-$ idempotent)
Hence, S is commutative. Also we have for every $a,b,c\in S$ and $\gamma\in\Gamma$. Then,
$(a\gamma b)\beta c=(c\gamma b)\beta a$ (by left invertive law)
$=a\beta(b\gamma c)$ (by commutative law)
$=a\gamma(b\beta c)$ (by lemma 1.4.)

Hence, S is Γ-semigroup. Now we should prove e is right identity. Then for every
$a \in S$ and $\gamma \in \Gamma$ we have,

$a\gamma e = (a\gamma a)\gamma e$ (by $\Gamma -$ idempotent)

$= (e\gamma a)\gamma a$ (by left invertive law)

$= a\gamma a = a$ (by $\Gamma -$ idempotent)

Hence, S is a Γ-semigroup with identity e i.e. S is Γ- monoid.

Conclusion. This current article inveatigates the ideas of T^1, T^2, T^3 and $T^4 - \Gamma - AG$- groupoids. By theorems and propositions we inveatigate that every $T^4 - \Gamma - AG$- groupoid with left identity is $\Gamma -$ paramedical, Left nuclear square and $T^1 - \Gamma - AG$-groupoid are $T^3 - \Gamma - AG$- groupoid.So every $T^2 - \Gamma - AG$- groupoid is $\Gamma -$ paramedical and $T^2 - \Gamma - AG$- groupoid with left identity is left nuclear square and every $T^1 - \Gamma - AG$-groupoid with left identity is left nuclear square and every $Bol^* - \Gamma - AG$-groupoid with left identity is $\Gamma -$ paramedical and left nuclear square.

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