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Some Properties for A Portfolio Optimization Model

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Abstract

Portfolio optimization Problem is to find the securities portfolio minimizing the risk for a required return or maximizing the return for a given risk level.In this paper, we discuss a portfolio investment model with expected rate of return under non-negative constrains.we proved some properties of the model.Using these properties, the model solving will be simplified.

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1 Introduction

Portfolio theory has been an important part of modern finance theory. The traditional portfolio optimization problem is to find an investment plan for securities with a reasonable trade-off between the rate of return and risk.The mean-variance model of Markowitz is designed to obtain the portfolio which can achieve a specified average rate of return with the minimum risk[1]. Because of its computational complexity, many researchers study the method to solve the model[2,3].The main aim of the present paper is to give some properties for the Markowitz model so as to simplified the solving process.

2 Mathematical Model

The following model is based upon mean-variance model by Markowitz[1].

$$\begin{cases} Min \quad f(w) = w^T V w \\ s.\ t. \quad Aw = b \\ \qquad\quad w \geq 0 \end{cases} \tag{1}$$

Where

$$w = (w_1, x_2, \ldots, w_n)^T, \quad V = (\sigma_{ij})_{n\times n}$$

$$A = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \; b = \begin{pmatrix} r^* \\ 1 \end{pmatrix},$$

n is the number of securities selected by investors;
w_i is the proportion of securities i in the portfolio;
$w = (w_1, w_2, \ldots, w_n)$ is the investment portfolio;
σ_{ij} is the covariance between securities i and j;
$f(w)$ is the risk measured by the variance of the portfolio;
r_i is the expected return rate of securities i;
r^* is the expected return rate of the portfolio.

If we remove the constrains $w \geq 0$, then we get the model (2):

$$\begin{cases} Min \quad f(w) = w^T V w \\ s.\ t. \quad Aw = b \end{cases} \tag{2}$$

3 Some Properties for Mean-Variance Model

Model (1) and (2) are both quadratic programming model, for the covariance matrix V be positive semi-definite, the optimal solutions of model (1) and (2) are existed and unique.For simplicity we further assume the matrix V be positive definite, and the optimal solution is of uniqueness.[2]

We denotes the optimal solution of model (1) by $w^{(1)} = (x_1, x_2, \ldots, x_n)^T$, and denotes the optimal solution of model (2) by $w^{(2)} = (y_1, y_2, \ldots, y_n)^T$ .Now we describe some useful properties for the two models.

Lemma 1[3] $w^{(2)}$ be the optimal solution of model (2), if and only if there exists a vector α satisfy

$$\begin{cases} 2Vw^{(2)} = A^T \alpha \\ Aw^{(2)} = b \end{cases}$$

and the optimal solution of model (2) is $w^{(2)} = V^{-1}A^T(AV^{-1}A^T)^{-1}b$.

Lemma 2[3] $w^{(1)}$ be the optimal solution of model (1), if and only if there exists a vector β and a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ satisfy

$$\begin{cases} 2Vw^{(1)} = A^T \beta + \gamma \\ Aw^{(1)} = b \\ \gamma_i x_i = 0, \quad i = 1, 2, \ldots, n \\ w^{(1)} \geq 0, \quad \gamma \geq 0 \end{cases}$$

Theorem 1 If $w^{(1)} \neq w^{(2)}$, then there exists index i satisfy $x_i = 0$ and $y_i < 0$.

Proof. Because $w^{(1)} \neq w^{(2)}$, we have $f(w^{(1)}) > f(w^{(2)})$, and there is at least one index i satisfy $y_i < 0$. Define $I = \{i | y_i < 0\} \neq \phi$.

Using Lemma 1 and Lemma 2 , by calculation it is easy to get

$$\sum_{i=1}^{n} y_i \gamma_i = \sum_{i \in I} y_i \gamma_i + \sum_{i \notin I} y_i \gamma_i < 0.$$

Since $\sum_{i \notin I} y_i \gamma_i \geq 0$, therefore $\sum_{i \in I} y_i \gamma_i < 0$.

Thus there is at least one index $i \in I$ makes $\gamma_i > 0$, and further from Lemma 2, we know that the corresponding $x_i = 0$.

For $x_i = 0$, we call the securities i is superfluous for model (1).

Without loss of generality, we assume the vector γ in Lemma 2 satisfy $\gamma_i = 0 \quad (i = 1, 2, \ldots, m)$ and $\gamma_i > 0 \quad (i = m+1, m+2, \ldots, n)$.
Accordingly, denote the optimal solution of model (1) by $w^{(1)} \equiv ((x^{(1)})^T, (x^{(2)})^T)^T$, where $x^{(1)} = (x_1, x_2, \ldots, x_m)^T$, $x^{(2)} = (x_{m+1}, x_{m+2}, \ldots, x_n)^T$. Then we have the following conclusions.

Theorem 2 Let $A_1 = \begin{pmatrix} r_1 & r_2 & \cdots & r_m \\ 1 & 1 & \cdots & 1 \end{pmatrix}$, $V_{11} = (\sigma_{ij})_{m \times m}$,then $x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0; x_{m+1} = 0, x_{m+2} = 0, \ldots, x_n = 0$; and $x^{(1)} = (x_1, x_2, \ldots, x_m)^T$ is the optimal solution of the following model (3)

$$\begin{cases} Min \;\; f_1(w_1, w_2, \ldots, w_m) = (w_1, w_2, \ldots, w_m) V_{11} (w_1, w_2, \ldots, w_m)^T \\ s.\,t. \;\; A_1 (w_1, w_2, \ldots, w_m)^T = b \end{cases} \qquad (3)$$

Proof. According to Lemma 2, the $x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0; x_{m+1} = 0, x_{m+2} = 0, \ldots, x_n = 0$ is evident, and $2Vw^{(1)} = A^T\beta + \gamma$ and $Aw^{(1)} = b$, through a simple calculation we know that

$$2V_{11}x^{(1)} = A_1^T\beta \text{ and } A_1x^{(1)} = b \qquad .$$

It is obtained from lemma 1 that $x^{(1)} = (x_1, x_2, \ldots, x_m)^T$ is the optimal solution of model (3).

Theorem 3 When the number of securities in the portfolio is reduced, the risk of the optimal portfolio will not fall.

Proof. Without loss of generality, we consider model (2) and model (3)($0 < m < n$).Let $w^{(2)} = (y_1, y_2, \ldots, y_n)^T$ be the optimal solution of model (2), $w^{(3)} = (\bar{y_1}, \bar{y_2}, \ldots, \bar{y_m})^T$ be the optimal solution of model (3).
Because $((w^{(3)})^T, 0)^T$ is the feasible solution of model (2), so we have
$f(w^{(2)}) = (w^{(2)})^T V w^{(2)} \leq ((w^{(3)})^T, 0) V ((w^{(3)})^T, 0)^T = (w^{(3)})^T V_{11} w^{(3)} = f_1(w^{(3)})$.

Remark: Theorem 2 and Theorem 3 show that, remove the superfluous securities of model (1) from model (2),the simplified model (2) [That is model (3), and without regard the non-negative of w_i] can contribute the optimal solution of model (1).

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