Some Inequalities for m-Convex Stochastic Process

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ABSTRACT

In this paper we present some Hermite-Hadamard and Fejér type inequalities as counterpart of the developed for functions. We generalize results given for m$-$convex functions in Bracamontes, Giménez, Merentes, & Vivas, 2016, Dragomir & Toader, Some inequalities for m-convex functions, 1993 and Özdemir, Avci, & Set, 2010 among them, right-hand side of Hermite-Hadamard type and Fejér type inequalities.

Keywords: Convex; m$-$convex stochastic processes; Fejér type inequality; Hermite-Hadamard type inequality; Convex analysis

Introduction

Many inequalities have been established for convex functions and one of most famous is the Hermite-Hadamard inequality, due to the rich geometrical significance and applications see Fejér (1906), Niculescu & Persson (2006).


In 1993, S. S. Dragomir and Gh. Toader Dragomir & Toader, Some inequalities for m-convex functions, 1993 demonstrated the Hermite-Hadamard inequality for functions whose absolute values of one of most famous is the Hermite-Hadamard inequality, due to the rich geometrical significance and applications see Fejér (1906), Niculescu & Persson (2006).

Some inequalities for m-convex functions were developed by M.K. Bakula et al. Bacula, Pečarić, & Ribičić, (2006). Also, in 2010 M. E. Özdemir et al. Özdemir, Avci, & Set, 2010 gave some estimates to the right-hand of Hermite-Hadamard inequality for functions whose absolute values of second derivatives raised to positive real power are m$-$convex.

On the order hand, in the same year, Bo-Yan Xi et al. introduce concepts of the m$-$convex and $(\alpha,m)$-geometrically convex and establish some inequalities of Hermite-Hadamard type for these classes of functions. Xi, Bai & Qi, 2012.


Recently, S. Özcan in Özcan, 2019 introduced the concepts of m$-$convex and $(\alpha,m)$-convex stochastic process, as well as some Hermite-Hadamard type inequalities for the first derivative were established.

In this paper, some Hermite-Hadamard and Fejér type inequalities of m$-$convex functions for m$-$convex presented in Dragomir, 2002, Lara, Rosales, & Sánchez, 2015 and Özdemir, Avci, & Set, 2010 are develop.

PRELIMINARIES

Let $(\Omega,A,P)$ be a probability space. A function $Y:\Omega \rightarrow \mathbb{R}$ is a random variable if it is $A$-measurable. A function $X:\Omega \rightarrow \mathbb{R}$, where $1 \in \mathbb{R}$ is an interval, is a stochastic process if for every $t \in \mathbb{I}$ the function $X(t)$ is a random variable.

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Received: November 27, 2019; Accepted: February 11, 2020; Published: February 18, 2020

Citation: Lara T, Mejia O, Merentes N, Valera Lopez M (2020) Some inequalities for m$-$convex stochastic process. Mathematica Eterna. 10: 103. 10.35248/1314-3344.20.10.103.

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Definition 2.1. A stochastic process X: I×Ω→R is:
1) Jensen-convex if, for every a,b ∈ I the following inequality is satisfied:
\[ x\left(\frac{a+b}{2}\right) \leq \frac{x(a)+x(b)}{2} \quad (a.e.) \] (1)
2) convex if, for every a,b ∈ I, t ∈ (0,1), the following inequality is taken place:
\[ x(ta+(1-t)b) \leq tx(a)+(1-t)x(b) \quad (a.e.) \] (2)
Also, we say that a stochastic process X: I×Ω→R is:
1) continuous in probability in the interval I, if for all t ∈ I we have
\[ \lim_{t \to t_0} E\left[X(t)\right] = X(t_0) \]
Where P-\lim denotes the limit in probability.
2) Mean-square continuous in the interval I, if for all t ∈ I we have
\[ \lim_{t \to t_0} E\left[|X(t) - X(t_0)|^2\right] = 0 \]
3) mean-square differentiable at a point t ∈ I if there is a random variable X'(t): I×Ω→R:
\[ X'(t) = P-\lim_{t \to t_0} \frac{X(t) - X(t_0)}{t - t_0} \]
Note that mean-square continuity implies continuity in probability, but the converse is not true.

Example 2.2. Let X: I×Ω→R be a stochastic process with E[X(t)^2] < ∞ for all t ∈ I, a,b ∈ I, and Θk ∈ {t-(k-1),t_k} for all k=1,...,n, a random variable Y: Ω→R is called mean-square integral of the process X on [a,b], if for a normal sequence of partitions of the interval [a,b] and for all Θ_k ∈ {t-(k-1),t_k} for all k=1,...,n, we have:
\[ \lim_{k \to \infty} E\left[\sum_{i=1}^{n} X(\Theta_i)(t_i - t_{i-1})\right] = 0. \]
In such case, we write
\[ Y(t) = \int_{a}^{t} X(s)ds \quad (a.e.) \]
For the existence of the mean-square integral is enough to consider the mean-square continuity of the stochastic process X. Basic properties of the mean-square integral can be read in (Sobczyk, 1991).

Now, in (Özcan, 2019) was introduced the definition of m-convex stochastic process.

Definition 2.2. The mean-square stochastic process X: I×Ω→R is said to be m-convex, if for every a,b ∈ I and t ∈ [0,1], we have:
\[ X(ta + mb) \leq tx(a) + mX(b) \quad (a.e.) \] (3)
Denote by S.m (c) the class of the m-convex stochastic process on I×Ω for which X(0)=0.

Remark 2.3. From the Definition 2.2. We have the following immediate results:
1. If a,b=0 then X(0)=0
2. For m=1, we recapture the concept of convex stochastic process (Nikodem, 1980) defined on I×Ω and for m=0, we get the concept of starshaped stochastic process on I×Ω. We recall that X: I×Ω→R is starshaped if
\[ X(ta) \leq tx(a) \quad (a.e.) \] (4)
for all t ∈ (0,1)\[a,b\].

Due to the Remark 2.3. you have the following lemma:

Lemma 2.4. If X is in the class S.m(c), then it is starshaped.
Proof. For any a ∈ I and t ∈ (0,1], we have:
\[ X(ta) = X(ta + m(1-t)0) \leq tx(a) + m(1-t)X(0) \leq tx(a) \quad (a.e.) \] (5)
Almost everywhere.

Lemma 2.5. If X is a m-convex stochastic process and S.mSnS1, then X is m-convex.
Proof. If a,b ∈ I and t ∈ (0,1], then:
\[ X(ta + mb) = X(ta + m(1-t)b) \leq tx(a) + m(1-t)X(b) \]
= tx(a) + m(1-t)X(b) Almost everywhere and the lemma are proved.

MAIN RESULT
In order to prove the Hermite-Hadamard inequality for m-convex stochastic processes we establish the following results.

Theorem 3.1. Let X: I×Ω→R be a stochastic process non negative, m-convex mean-square integrable stochastic process, with m ∈ [0,1]. For every a,b ∈ I, a < b the following inequality is satisfied almost everywhere:
\[ X\left(\frac{a+mb}{2}\right) \leq \frac{1}{mb-a} \int_{a}^{b} X(S)ds \leq \frac{X(a) + mx(b)}{2} \] (6)
Proof. Let us calculate the right-hand side of (6). Since X is a m-convex stochastic process, we have:
\[ \int_{a}^{b} X(ta + mb)dt \leq \int_{a}^{b} tx(a)dt + \int_{a}^{b} mX(b)dt \]
= \frac{X(a) + mx(b)}{2}
Making a change of variables t = ta + mb, in the integral:
\[ \int_{a}^{b} X(ta + mb)dt = \frac{1}{a - mb} \int_{a}^{b} X(S)ds \]
= \frac{1}{a - mb} \int_{a}^{b} X(S)ds, \quad (a.e.)
The right-hand side of inequality (6) is obtained.
On the other hand, to demonstrate the left side of the inequality (6), the following transformation is performed:
\[ \frac{1}{mb-a} \int_{a}^{b} X(S)ds = \frac{1}{mb-a} \left[ \int_{\frac{a}{mb-a}X}^{mb} X(s)ds + \int_{mb}^{\frac{b}{mb-a}X} X(s)ds \right] \quad (a.e.) \]
Making the change of variable \[ s = \frac{mb + a + tmb - a}{2} \] and \[ s = \frac{mb + a - tmb - a}{2} \]
Proof. By the \( m \)-convexity of the stochastic process \( X \), we have that

\[
X\left(\frac{u+v}{2}\right) \leq \frac{1}{2} X(u) + mX\left(\frac{v}{m}\right)
\]

For all \( u, v \in I \) (a.e.)

If we choose \( u=ta+(1-t)b, v=(1-t)a+tb \), we deduce:

\[
X\left(\frac{a+b}{2}\right) \leq \frac{1}{2} X(ta+(1-t)b)dt + mX\left(\frac{(1-t)a+tb}{m}\right)
\]

For all \( t \in [0,1] \), taking into account that:

\[
\int_0^1 X(ta+(1-t)b)dt = \frac{1}{b-a} X(S)ds, \text{(a.e)}
\]

Almost everywhere for all \( t \in [0,1] \).

We deduce from (8) that

\[
X\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ \int \left( X(ta+(1-t)b)dt + mX\left(\frac{(1-t)a+tb}{m}\right)\right) \right]
\]

By the \( m \)-convexity of the stochastic process \( X \), from Definition 2.2, we have the following immediate results:

\[
\frac{1}{2} \left[ X(ta+(1-t)b)dt + mX\left(\frac{(1-t)a+tb}{m}\right)\right]
\]

\[
\leq X(a) + mX\left(\frac{b}{m}\right)
\]

\[
tX(a) + mX\left(\frac{b}{m}\right) + m(1-t)X\left(\frac{a}{m}\right)
\]

\[
+ m^2 (1-t)X\left(\frac{b}{m^2}\right)
\]

Almost everywhere, for all \( t \in [0,1] \).

Integrating (9) over \( t \in [0,1] \), we deduce:

\[
\frac{1}{b-a} \int X(u) + mX\left(\frac{u}{m}\right)\right)du \leq \frac{1}{2} \left[ X(a) + mX\left(\frac{b}{m}\right) + mX\left(\frac{a}{m}\right) + m^2 X\left(\frac{b}{m^2}\right)\right]
\]

Almost everywhere. By similar argument we can state:

\[
\frac{1}{b-a} \int X(u) + mX\left(\frac{u}{m}\right)\right)du \leq \frac{1}{2} \left[ X(a) + mX\left(\frac{b}{m}\right) + mX\left(\frac{a}{m}\right) + m^2 X\left(\frac{b}{m^2}\right)\right]
\]

And the proof is completed.

In order to prove the following inequalities, we need lemma bellow, demonstrated in (Barraez, Gonzalez, Merentes, & Motros, 2015).

Theorem 3.3. Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be a mean-square stochastic process on \( I \), \( a,b \in I \) with \( a \leq b \) and \( I=[0,\infty) \). If \( X \) is \( m \)-convex stochastic process \( m \in (0,1) \), then one has the inequality:

\[
X\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \int \left( X(ta+(1-t)b)dt + mX\left(\frac{(1-t)a+tb}{m}\right)\right)
\]

Theorem 3.2. Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be a mean-square stochastic process on \( I \), \( a,b \in I \) with \( a \leq b \) and \( I=[0,\infty) \). If \( X \) is \( m \)-convex stochastic process \( m \in (0,1) \), then one has the inequality:

\[
X\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \int \left( X(ta+(1-t)b)dt + mX\left(\frac{(1-t)a+tb}{m}\right)\right)
\]
Lemma 3.4. Let \( X: I × \Omega \rightarrow \mathbb{R} \) be a stochastic process mean-square differentiable on \( I, a,b\in I \) with \( a< b \). If \( X(t) \) is mean-square integrable on \( [a,b] \), then the following equality holds almost everywhere:

\[
\frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt = \frac{b-a}{2} \left( \int_0^1 (t-1)X'(ta+(1-t)b) dt \right)
\]

Theorem 3.5. Let \( X: I × \Omega \rightarrow \mathbb{R} \) be a stochastic process mean-square differentiable on \( I, a,b\in I \) with \( a< b \). If \( X(t) \) is a \( m \)-convex stochastic process, then the following inequality holds almost everywhere:

\[
\left( \frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt \right)^{q} \leq \frac{(b-a)^2}{2} \left( \int_0^1 (t-1)X'(ta+(1-t)b) dt \right)^2
\]

Proof. First suppose that \( q=1 \). From Lemma 3.4. we have:

\[
\frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt \leq \frac{(b-a)^2}{2} \left( \int_0^1 (t-1)X'(ta+(1-t)b) dt \right)
\]

Since \( |X| \) is \( m \)-convex stochastic process we know that for any \( t \in [0,1] \):

\[
\int (t-1)X'(ta+(1-t)b) dt \leq t \left[ X(a) + m(1-t) \right] \left( \int \left( \frac{b}{m} \right) dt \right)
\]

Therefore,

\[
\int \left( \frac{b}{m} \right) dt \leq \frac{(b-a)^2}{2} \left( \int_0^1 \left[ X(a) + m(1-t) \right] \left( \frac{b}{m} \right) dt \right)
\]

\[
= \frac{(b-a)^2}{2} \left[ \left( \frac{b}{m} \right) \int_0^1 \left[ X(a) + m(1-t) \right] \left( \frac{b}{m} \right) dt \right]
\]

\[
= \frac{(b-a)^2}{12} \left[ \left( \frac{b}{m} \right) \int_0^1 \left[ X(a) + m(1-t) \right] \left( \frac{b}{m} \right) dt \right]
\]

Almost everywhere, which complete the proof for this case.

Suppose now that \( q > 1 \). Using Lemma 3.4. and the Hölder’s inequality for \( q=p=q/(q-1) \), we obtain:

\[
\left[ \int (t-1)X'(ta+(1-t)b) dt \right]^{q} \leq \frac{(b-a)^2}{2} \left[ \int_0^1 (t-1)X'(ta+(1-t)b) dt \right]^{2q/(q-1)}
\]

Hence, from (10) and (11) we obtain:

\[
\left( \frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt \right)^{q} \leq \frac{(b-a)^2}{2} \left( \int_0^1 (t-1)X'(ta+(1-t)b) dt \right)^2
\]

Remark 3.6. If in Theorem 3.5 we choose \( m=1 \) and if \( \mathbb{X}(t_{-}) \leq K \), which complete the proof.

Theorem 3.7. Let \( X: I × \Omega \rightarrow \mathbb{R} \) be a stochastic process mean-square differentiable on \( I, a,b\in I \) with \( a< b \). If \( |X|^{p} \) is a \( m \)-convex stochastic process for some fixed \( q>1 \) and \( m \in (0,1] \), then the following inequality holds:

\[
\left( \frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt \right)^{q} \leq \frac{(b-a)^2}{2} \left( \int_0^1 \left[ X(a) + m(1-t) \right] \left( \frac{b}{m} \right) dt \right)^{2q/(q-1)}
\]

Almost everywhere, where \( p=q/(q-1) \).

Proof. From Lemma 3.4. and using the well-know Hölder’s inequality we have successively almost everywhere:

\[
\left( \frac{X(a)+X(b)}{2} - \frac{1}{b-a} \int_a^b X(t) dt \right)^{q} \leq \frac{b-a^2}{2} \left[ \int_0^1 (t-1)X'(ta+(1-t)b) dt \right]
\]

\[
+ \frac{m}{2} \left[ \int_0^1 \left( \frac{b}{m} \right) dt \right]^{q} \leq \frac{(b-a)^2}{2} \left[ \int_0^1 \left[ X(a) + m(1-t) \right] \left( \frac{b}{m} \right) dt \right]^{2q/(q-1)}
\]

Almost everywhere, where \( 1/p+1/q=1 \). We note that, the Beta and Gamma function

\[
\beta(x,y) = \frac{1}{\Gamma(x)\Gamma(y)} \int_0^1 t^{x-1}(1-t)^{y-1} dt, \; x,y>0,
\]

\[
r(x) = \frac{1}{\Gamma(x)} \int_0^\infty t^{x-1}e^{-t} dt, x\geq 0,
\]
And used to evaluate the integral
\[
\int_0^1 (t - i)^r dt = \int_0^1 (t - 1)^y dt = \beta(p+1,p+1),
\]
Where
\[
\beta(p+1,p+1) = 2^{3(p-1)} \left( \frac{1}{2} \right)^r \left( \frac{p+1}{r+1} \right)^2.
\]
and \(\Gamma(1/2) = \sqrt{\pi}\), which completes the proof.

**Corollary 3.8.** With the above assumptions given that \(|X^*(t,\cdot)| < K on [a,b]\) and \(0 < s \leq 1\), we have the inequality almost everywhere:
\[
\frac{X(a) + X(b)}{2} \leq - \frac{1}{b-a} \int X(t) dt \leq \min \{k, k_2\}, (a.e)
\]
Where
\[
k_i = \frac{(b-a)^2}{12} \left[ \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \right]^{\frac{1}{q}}, (a.e)
\]
and
\[
k_2 = \frac{(b-a)^2}{8} \left[ \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \right]^{\frac{1}{q}}, (a.e)
\]

**Theorem 3.10.** With the assumptions of Theorem 3.7 we have the following inequality almost everywhere:
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \frac{1}{2} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \int (1-t)^{q\gamma} dt\left(1 \right) + m(1+q) \int \left( \frac{b}{m} \right)^{q\gamma} dt\right)^{\frac{1}{q}}, (a.e)
\]
Almost everywhere. Since, \(\lim_{r \to 1} \left( \frac{1}{2} \right)^r = 1\) and \(\lim_{r \to 1} \left( \frac{1}{2} \right)^r = \frac{1}{2}\). We have,
\[
\left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r < 1, q \in (1,\infty).
\]
Hence, for \(q\in(1,\infty)\),
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \frac{1}{2} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \int (1-t)^{q\gamma} dt\left(1 \right) + m(1+q) \int \left( \frac{b}{m} \right)^{q\gamma} dt\right)^{\frac{1}{q}},
\]
almost everywhere.

**Theorem 3.11.** Let \(X: \Omega \to \mathbb{R}\) be a stochastic process mean-square differentiable on \(I\), a, b \( \in \mathbb{R}\) with \(a < b\). If \(|X'|^{q}\) is a \(m\)-convex stochastic process for some fixed \(q > 1\) and \(m \in (0,1]\), then the following inequality holds:
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \frac{1}{2} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \int (1-t)^{q\gamma} dt\left(1 \right) + m(1+q) \int \left( \frac{b}{m} \right)^{q\gamma} dt\right)^{\frac{1}{q}},
\]
almost everywhere.

**Proof.** From Lemma 3.4. and the well-known power-mean inequality, we obtain:
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \frac{1}{2} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \int (1-t)^{q\gamma} dt\left(1 \right) + m(1+q) \int \left( \frac{b}{m} \right)^{q\gamma} dt\right)^{\frac{1}{q}}
\]
Almost everywhere. Since, \(\lim_{r \to 1} \left( \frac{1}{2} \right)^r = 1\) and \(\lim_{r \to 1} \left( \frac{1}{2} \right)^r = \frac{1}{2}\),
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \min \{E_i, E_{i+1}\}, (a.e)
\]
With completes the proof.

**Remark 3.12.** From Theorem 3.7. – 3.11., we have:
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \frac{2}{(q+1)(q+2)\gamma} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^r \int (1-t)^{q\gamma} dt\left(1 \right) + m(1+q) \int \left( \frac{b}{m} \right)^{q\gamma} dt\right)^{\frac{1}{q}}
\]
Almost everywhere. 

Since \(\left( \frac{2}{(q+1)(q+2)\gamma} \right)^{\frac{1}{q}} \leq 1\), \(q \in [1,\infty)\) we obtain
\[
\frac{X(a) + X(b)}{2} - \frac{1}{b-a} \int X(t) dt \leq \min \{E_i, E_{i+1}\}, (a.e)
\]
With completes the proof.
In Fejér, 1906, L. Fejér gives a generalization of (6). Now, we shall present the definition of Fejér inequality for convex stochastic processes.

**Theorem 3.13.** Let \( X: I \times \Omega \rightarrow \mathbb{R} \) be a non-negative convex mean-square integrable stochastic process. For every \( a < b \) with \( a < b \), the following inequality is satisfied almost everywhere:

\[
X\left(\frac{a+b}{2}\right)Y(t)dt \leq \frac{1}{b-a}\int_a^b X(t)Y(t)dt
\]

(12)

Where \( Y: \Omega \rightarrow (0, \infty) \) is a non-negative mean-square integrable stochastic process, symmetric with respect to \((a+b)/2\) that is, \( Y(a+b-t, v) = Y(t, v) \).

The following establishes some results that represent the counterpart of the results presented by M. Bracamontes et al. in (Bracamontes, Giménez, Merentes, & Vivas, 2016) for m-convex stochastic processes:

**Theorem 3.14.** Let \( X: [0, \infty) \times \Omega \rightarrow R \) be a m-convex mean-square integrable stochastic process with \( m \in [0,1] \). For every \( a, b \in \mathbb{R} \) with \( a < b \), and \( Y: [a, b] \times \Omega \rightarrow \mathbb{R} \) is a non-negative mean-square integrable stochastic process, symmetric with respect to \((a+b)/2\), then the following inequality is satisfied almost everywhere:

\[
X\left(a\right)X\left(b\right) \geq \frac{1}{b-a}\int_a^b X(t)Y(t)dt
\]

(13)

**Proof.** Since \( Y \) is a non-negative mean-square integrable stochastic processes on \([a, b] \times \Omega \) and symmetric with respect to \((a+b)/2\), then the following inequality is satisfied almost everywhere:

\[
\frac{1}{b-a}\int_a^b X(t)Y(t)dt \leq \frac{X(a) + X(b)}{2}\int_a^b \frac{b-t}{b-a} Y(t)dt
\]

Almost everywhere. Hence, the m-convexity of \( X \) implies:

\[
\frac{1}{b-a}\int_a^b X(t)Y(t)dt \leq \frac{X(a) + X(b)}{2}\int_a^b \frac{b-t}{b-a} Y(t)dt + \frac{m}{2}\int_a^b \frac{t-a}{b-a} Y(t)dt
\]

Which proves the result.

**Remark 3.15.** Notice that if we make \( m=1 \) in (13) we get the right-hand side of inequality (12) that is:

\[
\frac{b}{a}\int_a^b X(t)Y(t)dt \leq \frac{X(a) + X(b)}{2}\int_a^b Y(t)dt
\]

In the following, a bound is obtained for the right-hand side of inequality (12) for m-convex stochastic processes.

**Theorem 3.16.** Let \( X: [0, \infty) \times \Omega \rightarrow \mathbb{R} \) be a m-convex mean-square integrable stochastic process in \([a, b] \times \Omega \) with \( m \in (0,1) \). For every \( b \in (0, \infty) \) \( s \in (0, a) \), and \( X: [a, b] \times \Omega \rightarrow \mathbb{R} \) is a non-negative mean-square integrable stochastic process, symmetric with respect to \((a+b)/2\), then the following inequality is satisfied almost everywhere:

\[
X\left(\alpha X(a) + \beta X(b)\right) \geq \frac{a+b}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

where \( \alpha, \beta \) are real numbers with \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \).

**Proof.** The m-convexity of the stochastic process \( X \) implies that

\[
X\left(\alpha X(a) + \beta X(b)\right) \geq \frac{a+b}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

Almost everywhere. Now, \( X \) is symmetrical then:

\[
= \frac{1}{2}\int_a^b X\left(\alpha + \beta - t\right)Y(t)dt + \frac{m}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

Which proves the result.

**Remark 3.17.** If \( m=1 \) in Theorem 3.16 we obtain:

\[
X\left(\alpha X(a) + \beta X(b)\right) \geq \frac{1}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

Which is the left-hand side of (12).

Now, we present a generalization of (12). First, we prove the following result:

**Lemma 3.18.** Let \( X: [0, \infty) \times \Omega \rightarrow \mathbb{R} \) be a m-convex mean-square integrable stochastic process with \( m \in (0,1) \). For every \( \alpha, \beta \in (0, \infty) \), there is \( \alpha\neq \beta \) such that the following inequality holds almost everywhere:

\[
X\left(\alpha X(a) + \beta X(b)\right) \geq \frac{a+b}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

**Proof.** Since \( X \) can be written as \( \alpha X(a) + \beta X(b) \), for some \( \alpha\in[0,1] \) and \( \alpha+b=X(1-a) \), then we have:

\[
X\left(\alpha X(a) + \beta X(b)\right) = \frac{1}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt
\]

Almost everywhere. Hence, the m-convexity of \( X \) implies:

\[
\frac{1}{2}\int_a^b X\left(\frac{t}{m}\right)Y(t)dt \leq \frac{X(a) + X(b)}{2}\int_a^b \frac{b-t}{b-a} Y(t)dt
\]

(14)
Almost everywhere.

**Theorem 3.19.** Under the same hypotheses of Theorem 3.14, the following inequality holds almost everywhere:

\[
\int_a^b X(t)Y(t)\,dt \leq \left( \frac{m}{2} \left[ X\left( \frac{a}{m} \right) + X\left( \frac{b}{m} \right) \right] + \frac{X(a) + X(b)}{2} \right) \int_a^b Y(t)\,dt
\]

Proof. By the symmetry of Y with respect to \((a+b)/2\) and Lemma 3.18:

\[
\int_a^b X(t)Y(t)\,dt = \int_a^b X(a+b-t)Y(a+b-t)\,dt + \int_a^b X(t)Y(t)\,dt
\]

\[
= \int_a^b \left( \frac{1}{2} X(a+b-t)Y(t) + \frac{1}{2} X(t)Y(t) \right)\,dt
\]

\[
= \frac{1}{2} \int_a^b \left[ m(1-a) \left( X\left( \frac{a}{m} \right) + X\left( \frac{b}{m} \right) \right) + a \left( X(a) + X(b) \right) - X(t) \right] Y(t)\,dt
\]

\[
\leq \left( \frac{m}{2} \left[ X\left( \frac{a}{m} \right) + X\left( \frac{b}{m} \right) \right] + \frac{X(a) + X(b)}{2} \right) \int_a^b Y(t)\,dt
\]

Almost everywhere.

**Remark 3.20.** Notice that if \(m=1\) in Theorem 3.19, we indeed get:

\[
\int_a^b X(t)Y(t)\,dt \leq \left( X(a) + X(b) \right) \int_a^b Y(t)\,dt, (a.e)
\]

**REFERENCES**


