Solutions of the Pell Equations $x^2 - (a^2b^2 + 2b)y^2 = N$

when $N \in \{\pm 1, \pm 4\}$

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Abstract

Let $a$ and $b$ be natural number and $d = a^2b^2 + 2b$. In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the equations $x^2 - dy^2 = \pm 1$ and we get all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find all positive integer solutions of the equations $x^2 - dy^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

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1 Introduction

The quadratic Diophantine equation of the form $x^2 - dy^2 = 1$ where $d$ is a positive square-free integer is called a Pell Equation after the English mathematician John Pell. The equation $x^2 - dy^2 = 1$ has infinitely many solutions $(x, y)$ whereas the negative Pell equation $x^2 - dy^2 = -1$ does not always have a solution. Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$. Whether or not there exists a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$. It can be seen that the equation $x^2 - 15y^2 = -1$ has no positive integer solutions. To find all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$, one first determines a fundamental solution. In this paper, after the Pell equations are described briefly, the fundamental solution to the Pell equations $x^2 - (a^2b^2 + 2b)y^2 = \pm 1$ are calculated
by means of the convergent of continued fraction of \( \sqrt{a^2b^2 + 2b} \). Moreover, all positive integer solutions of \( x^2 - (a^2b^2 + 2b)y^2 = \pm 4 \) and \( x^2 - (a^2b^2 + 2b)y^2 = \pm 1 \) are given in terms of the generalized Fibonacci and Lucas sequences. Especially, all positive integer solutions of the equations \( x^2 - (k^2 + 2)y^2 = \pm 4 \) and \( x^2 - (k^2 + 2)y^2 = \pm 1 \) are discovered.

Now we briefly mention the generalized Fibonacci and Lucas sequences \( (U_n(k, s)) \) and \( (V_n(k, s)) \). Let \( k \) and \( s \) be two nonzero integers with \( k^2 + 4s > 0 \). Generalized Fibonacci sequence is defined by

\[
U_0(k, s) = 0, \quad U_1(k, s) = 1
\]

and

\[
U_{n+1}(k, s) = kU_n(k, s) + sU_{n-1}(k, s)
\]

for \( n \geq 1 \) and generalized Lucas sequence is defined by

\[
V_0(k, s) = 2, \quad V_1(k, s) = k
\]

and

\[
V_{n+1}(k, s) = kV_n(k, s) + sV_{n-1}(k, s)
\]

for \( n \geq 1 \), respectively. It is well known that

\[
U_n(k, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}
\]

and

\[
V_n(k, s) = \alpha^n + \beta^n
\]

where \( \alpha = (k + \sqrt{k^2 + 4s})/2 \) and \( \beta = (k - \sqrt{k^2 + 4s})/2 \). The above identities are known as Binet’s formula. Clearly, \( \alpha + \beta = k \), \( \alpha - \beta = \sqrt{k^2 + 4s} \), and \( \alpha\beta = -s \).

For more information about generalized Fibonacci and Lucas sequences, one can consult [14],[7],[13],[9] and [10].

2 Preliminary Notes

Let \( d \) be a positive integer which is not a perfect square and \( N \) be any nonzero fixed integer. Then the equation \( x^2 - dy^2 = N \) is known as Pell equation. For \( N = \pm 1 \), the equations \( x^2 - dy^2 = 1 \) and \( x^2 - dy^2 = -1 \) are known as classical Pell equation. If \( a^2 - db^2 = N \), we say that \( (a, b) \) is a solution to the Pell equation \( x^2 - dy^2 = N \). We use the notations \((a, b)\) and \( a + b\sqrt{d} \) interchangeably to denote solutions of the equation \( x^2 - dy^2 = N \). Also, if \( a \) and \( b \) are both positive, we say that \( a + b\sqrt{d} \) is a positive solution to the equation \( x^2 - dy^2 = N \). Among these there is a least solution \( a_1 + b_1\sqrt{d} \), in
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which \( a_1 \) and \( b_1 \) have their least positive values. Then the number \( a_1 + b_1\sqrt{d} \)
is called the fundamental solution of the equation \( x^2 - dy^2 = N \). Recall that
if \( a + b\sqrt{d} \) and \( r + s\sqrt{d} \) are two solutions to the equation \( x^2 - dy^2 = N \), then
\( a = r \) if and only if \( b = s \), and \( a + b\sqrt{d} < r + s\sqrt{d} \) if and only if \( a < r \) and
\( b < s \).

Continued fraction plays an important role in solutions of the Pell equations
\( x^2 - dy^2 = 1 \) and \( x^2 - dy^2 = -1 \). Let \( d \) be a positive integer that is not a
perfect square. Then there is a continued fraction expansion of \( \sqrt{d} \) such that
\( \sqrt{d} = [a_0, a_1, a_2, ..., a_{l-1}, 2a_0] \) where \( l \) is the period length and the \( a_j \)'s are given
by the recursion formulas;

\[
\alpha_0 = \sqrt{d}, a_k = \lfloor \alpha_k \rfloor
\]
and
\[
\alpha_{k+1} = \frac{1}{\alpha_k - a_k}, k = 0, 1, 2, 3, ...
\]

Recall that \( a_l = 2a_0 \) and \( a_{l+k} = a_k \) for \( k \geq 1 \). The \( n^{th} \) convergent of \( \sqrt{d} \) for
\( n \geq 0 \) is given by

\[
\frac{p_n}{q_n} = [a_0, a_1, ..., a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}
\]

By means of the \( k^{th} \) convergent of \( \sqrt{d} \), we can give the fundamental solution
of the equations \( x^2 - dy^2 = 1 \) and \( x^2 - dy^2 = -1 \).

If we know fundamental solution of the equations \( x^2 - dy^2 = \pm 1 \) and
\( x^2 - dy^2 = \pm 4 \), then we can give all positive integer solutions to these equations.
For more information about Pell equation, one can consult [12] and [15].

Now we give the fundamental solution of the equations \( x^2 - dy^2 = \pm 1 \) by
means of the period length of the continued fraction expansion of \( \sqrt{d} \).

**Lemma 2.1** Let \( l \) be the period length of continued fraction expansion of
\( \sqrt{d} \). If \( l \) is even, then the fundamental solution to the equation \( x^2 - dy^2 = 1 \)
is given by

\[
x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}
\]
and the equation \( x^2 - dy^2 = -1 \) has no integer solutions. If \( l \) is odd, then the
fundamental solution to the equation \( x^2 - dy^2 = 1 \) is given by

\[
x_1 + y_1\sqrt{d} = p_{2l-1} + q_{2l-1}\sqrt{d}
\]
and the fundamental solution to the equation \( x^2 - dy^2 = -1 \) is given by

\[
x_1 + y_1\sqrt{d} = p_{l-1} + q_{l-1}\sqrt{d}
\]
Theorem 2.2 Let \( x_1 + y_1 \sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = 1 \). Then all positive integer solutions of the equation \( x^2 - dy^2 = 1 \) are given by
\[
x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^n
\]
with \( n \geq 1 \).

Theorem 2.3 Let \( x_1 + y_1 \sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = -1 \). Then all positive integer solutions of the equation \( x^2 - dy^2 = -1 \) are given by
\[
x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^{2n-1}
\]
with \( n \geq 1 \).

Now we give the following two theorems from [15]. See also [4].

Theorem 2.4 Let \( x_1 + y_1 \sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = 4 \). Then all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by
\[
x_n + y_n \sqrt{d} = \frac{(x_1 + y_1 \sqrt{d})^n}{2^{n-1}}
\]
with \( n \geq 1 \).

Theorem 2.5 Let \( x_1 + y_1 \sqrt{d} \) be the fundamental solution to the equation \( x^2 - dy^2 = -4 \). Then all positive integer solutions of the equation \( x^2 - dy^2 = -4 \) are given by
\[
x_n + y_n \sqrt{d} = \frac{(x_1 + y_1 \sqrt{d})^{2n-1}}{4^{n-1}}
\]
with \( n \geq 1 \).

From now on, we will assume that \( k, a, b \) are positive integers. We give continued fraction expansion of \( \sqrt{d} \) for \( d = a^2b^2 + 2b \) and \( d = a^2b^2 + b \). The proofs of the following two theorems are easy and they can be found many text books on number theory as an exercise.

Theorem 2.6 Let \( d = a^2b^2 + 2b \). Then
\[
\sqrt{d} = [ab, a, 2ab].
\]

Theorem 2.7 Let \( d = a^2b^2 + b \). If \( b \neq 1 \) then
\[
\sqrt{d} = [ab, 2a, 2ab]
\]
and if \( b = 1 \) then
\[
\sqrt{d} = [a, 2a].
\]
Corollary 2.8 Let \( d = a^2b^2 + 2b \). Then the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is
\[
  x_1 + y_1 \sqrt{d} = a^2b + 1 + a\sqrt{d},
\]
and the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions.

Proof The continued fraction expansion of \( \sqrt{d} = a^2b^2 + 2b \) is 2 by Theorem 2.6. Therefore the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is \( p_1 + q_1 \sqrt{d} \) by Lemma 2.1. Since
\[
  \frac{p_1}{q_1} = \frac{ab + 1}{a} = \frac{a^2b + 1}{a},
\]
the proof follows. Moreover, the period length of continued fraction expansion of \( \sqrt{a^2b^2 + 2b} \) is always even by Theorem 2.6. Thus by Lemma 2.1, it follows that the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions.

Corollary 2.9 Let \( d = a^2b^2 + b \). Then the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is
\[
  x_1 + y_1 \sqrt{d} = 2a^2b + 1 + 2a\sqrt{d}.
\]
Moreover, when \( b \neq 1 \), the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions and when \( b = 1 \), the fundamental solution to the equation \( x^2 - dy^2 = -1 \) is \( x_1 + y_1 \sqrt{d} = a + \sqrt{d} \).

Proof When \( b \neq 1 \), the period length of the continued fraction expansion of \( \sqrt{a^2b^2 + b} \) is 2 by Theorem 2.7. Therefore the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is \( p_1 + q_1 \sqrt{d} \) by Lemma 2.1. Since
\[
  \frac{p_1}{q_1} = \frac{ab + 1}{2a} = \frac{2a^2b + 1}{2a},
\]
the proof follows. When \( b = 1 \), the period length of the continued fraction expansion of \( \sqrt{a^2 + 1} \) is 1 by Theorem 2.7. Therefore the fundamental solution to the equation \( x^2 - dy^2 = 1 \) is \( p_1 + q_1 \sqrt{d} \) by Lemma 2.1. Since
\[
  \frac{p_1}{q_1} = \frac{a + 1}{2a} = \frac{2a^2 + 1}{2a},
\]
the proof follows. Moreover, when \( b \neq 1 \), the period length of continued fraction expansion of \( \sqrt{a^2b^2 + b} \) is always even by Theorem 2.7. Thus, by Lemma 2.1, it follows that the equation \( x^2 - dy^2 = -1 \) has no positive integer solutions. When \( b = 1 \), it can be seen that the fundamental solution to the equation \( x^2 - dy^2 = -1 \) is \( a + \sqrt{d} \) by Lemma 2.1 and Theorem 2.7.
3 Main Results

Theorem 3.1 Let \( d = a^2b^2 + 2b \). Then all positive integer solutions of the equation \( x^2 - dy^2 = 1 \) are given by

\[
(x, y) = (V_n(2a^2b + 2, -1)/2, aU_n(2a^2b + 2, -1))
\]

with \( n \geq 1 \).

Proof By Corollary 2.8 and Theorem 2.2, all positive integer solutions of the equation \( x^2 - dy^2 = 1 \) are given by

\[
x_n + y_n\sqrt{d} = (a^2b + 1 + a\sqrt{d})^n
\]

with \( n \geq 1 \). Let \( \alpha = a^2b + 1 + a\sqrt{d} \) and \( \beta = a^2b + 1 - a\sqrt{d} \). Then \( \alpha + \beta = 2a^2b + 2 \), \( \alpha - \beta = 2a\sqrt{d} \) and \( \alpha\beta = 1 \). Therefore

\[
x_n + y_n\sqrt{d} = \alpha^n
\]

and

\[
x_n - y_n\sqrt{d} = \beta^n.
\]

Thus it follows that

\[
x_n = \frac{\alpha^n + \beta^n}{2} = V_n(2a^2b + 2, -1)
\]

and

\[
y_n = \frac{\alpha^n - \beta^n}{2\sqrt{d}} = a\frac{\alpha^n - \beta^n}{2a\sqrt{d}} = a\frac{\alpha^n - \beta^n}{\alpha - \beta} = aU_n(2a^2b + 2, -1)
\]

by (1) and (2). Then the proof follows. Now we give all positive integer solutions of the equations \( x^2 - (a^2b^2 + 2b)y^2 = \pm 4 \). Before giving all solutions of the equations \( x^2 - dy^2 = \pm 4 \), we give the following theorems from [5].

Theorem 3.2 Let \( d \equiv 2(\text{mod} 4) \) or \( d \equiv 3(\text{mod} 4) \). Then the equation \( x^2 - dy^2 = -4 \) has positive integer solution if and only if the equation \( x^2 - dy^2 = 1 \) has positive integer solutions.

Theorem 3.3 Let \( d \equiv 0(\text{mod} 4) \). If fundamental solution to the equation \( x^2 - (d/4)y^2 = 1 \) is \( x_1 + y_1\sqrt{d/4} \), then fundamental solution to the equation \( x^2 - dy^2 = 4 \) is \( (2x_1, y_1) \).

Theorem 3.4 Let \( d \equiv 1(\text{mod} 4) \) or \( d \equiv 2(\text{mod} 4) \) or \( d \equiv 3(\text{mod} 4) \). If fundamental solution to the equation \( x^2 - dy^2 = 1 \) is \( x_1 + y_1\sqrt{d} \), then fundamental solution to the equation \( x^2 - dy^2 = 4 \) is \( (2x_1, 2y_1) \).
Theorem 3.5 Let \( d = a^2b^2 + 2b \). Then the fundamental solution to the equation \( x^2 - dy^2 = 4 \) is
\[
x_1 + y_1\sqrt{d} = 2(a^2b + 1) + 2a\sqrt{d}.
\]

Proof Assume that \( b \) is even. Then \( d \equiv 0 \pmod{4} \). Let \( b = 2k \) for some \( k \in \mathbb{Z} \). Then \( d/4 = a^2k^2 + k \). Thus, by Corollary 2.9, it follows that the fundamental solution to the equation \( x^2 - (a^2k^2 + k)y^2 = 1 \) is \((2a^2k + 1, 2a)\). Then, by Theorem 3.3, the fundamental solution to the equation \( x^2 - dy^2 = 4 \) is \(2(a^2b + 1, 2a)\). Then the proof follows.

Theorem 3.6 Let \( d = a^2b^2 + 2b \). Then the equation \( x^2 - dy^2 = -4 \) has no positive integer solutions.

Proof Assume that \( b \) is odd. If \( a \) is odd, then \( d \equiv 3 \pmod{4} \) and if \( a \) is even, then \( d \equiv 2 \pmod{4} \). Thus, by Theorem 3.2 and Corollary 2.8, it follows that the equation \( x^2 - dy^2 = -4 \) has no positive integer solutions. Assume that \( b \) is even and \( m^2 - dn^2 = -4 \) for some positive integer \( m \) and \( n \). Then \( d \) is even and therefore \( m \) is even. Let \( b = 2k \). Then
\[
m^2 - (4a^2k^2 + 4k)n^2 = -4
\]
and this implies that
\[
(m/2)^2 - (a^2k^2 + k)n^2 = -1.
\]
This is impossible by Corollary 2.9. Then the proof follows.

Theorem 3.7 All positive integer solutions of the equation \( x^2 - (a^2b^2 + 2b)y^2 = 4 \) are given by
\[
(x, y) = (V_n(2a^2b + 2, -1), 2abU_n(2a^2b + 2, -1))
\]
with \( n \geq 1 \).

Proof By Theorem 3.5, the fundamental solution to the equation \( x^2 - (a^2b^2 + 2b)y^2 = 4 \) is \(2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b} \). Therefore, by Theorem 2.4, all positive integer solutions of the equation \( x^2 - dy^2 = 4 \) are given by
\[
x_n + y_n\sqrt{d} = \frac{(2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b})^n}{2^{n-1}} = 2((2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b}/2)^n).
Let $\alpha = (2a^2b + 2 + 2a\sqrt{a^2b^2 + 2b})/2$ and $\beta = (2a^2b + 2 - 2a\sqrt{a^2b^2 + 2b})/2$. Then $\alpha + \beta = 2a^2b + 2$, $\alpha - \beta = 2a\sqrt{d}$ and $\alpha\beta = 1$. Thus it is seen that

$$x_n + y_n\sqrt{d} = 2\alpha^n$$

and

$$x_n - y_n\sqrt{d} = 2\beta^n.$$  

Therefore we get

$$x_n = \alpha^n + \beta^n = V_n(2a^2b + 2, -1)$$

and

$$y_n = \frac{\alpha^n - \beta^n}{\sqrt{d}} = 2a\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2aU_n(2a^2b + 2, -1)$$

by (1) and (2). Then the proof follows.

Let $a = k$ and $b = 1$. Then $d = a^2b^2 + 2b = k^2 + 2$. Thus we can give the following corollaries.

**Corollary 3.8** Let $d = k^2 + 2$. Then

$$\sqrt{k^2 + 2} = \lfloor k, k, 2k \rfloor.$$

**Corollary 3.9** Let $d = k^2 + 2$. Then the fundamental solution to the equation $x^2 - dy^2 = 1$ is

$$x_1 + y_1\sqrt{d} = k^2 + 1 + k\sqrt{d}.$$  

and the equation $x^2 - dy^2 = -1$ has no positive integer solutions.

**Corollary 3.10** Let $d = k^2 + 2$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by

$$(x, y) = (V_n(2k^2 + 2, -1)/2, kU_n(2k^2 + 2, -1))$$

with $n \geq 1$.

**Corollary 3.11** All positive integer solutions of the equation $x^2 - (k^2 + 2)y^2 = 4$ are given by

$$(x, y) = (V_n(2k^2 + 2, -1), 2kU_n(2k^2 + 2, -1))$$

with $n \geq 1$ and the equation $x^2 - (k^2 + 2)y^2 = -4$ has no positive integer solution.

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