Mathematica Aeterna, Vol. 5, 2015, no. 1, 95 - 104

Remarks on Jensen, Hermite-Hadamard and Fejer inequalities for strongly convex stochastic processes.

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Abstract

Discrete Jensen inequality for strongly midconvex stochastic processes and integral Jensen inequality for strongly convex stochastic processes are proved. Fejer inequality and the converse of Hermite–Hadamard theorem for strongly convex stochastic processes are presented.

Mathematics Subject Classification: 26A51, 26D15, 39B62, 60G99

Keywords: strongly convex stochastic processes, Jensen inequality, Fejer inequality

1 Introduction

In 1966 B.T. Polyak introduced the notion of strongly convex functions. Such functions play an important role in optimization theory and mathematical economies (see, for instance [8], [9], and the references therein). In 1980 K. Nikodem investigated properties of convex stochastic processes [7]. Later, A. Skowronski described the properties of Jensen-convex stochastic processes in [10]. Next, the Hermite-Hadamard inequality for convex and strongly stochastic processes was proved in [2] and [3]. In this article we will present some counterparts of Jensen and Fejer inequalities and we will show the converse theorem to Hermite-Hadamard's theorem.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a *random variable*, if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$ is called a *stochastic process*, if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $C : \Omega \to \mathbb{R}$ be a positive random variable. Recall, that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is *strongly convex with modulus* $C(\cdot)$, if

$$X\big(\lambda u + (1-\lambda)v, \cdot\big) \leqslant \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) - C(\cdot)\lambda(1-\lambda)(u-v)^2 \quad \text{(a.e.)} \quad (1)$$

for all $\lambda \in [0,1]$ and $u, v \in I$.

We say, that a stochastic process is *strongly Jensen–convex (or strongly mid-convex) with modulus* $C(\cdot)$, if the inequality (1) is assumed only for $\lambda = \frac{1}{2}$ and all $u, v \in I$, i.e.

$$X\big(\frac{u+v}{2}, \cdot\big) \leqslant \frac{1}{2}X(u,\cdot) + \frac{1}{2}X(v,\cdot) - \frac{C(\cdot)}{4}(u-v)^2 \quad \text{(a.e.)}. \tag{2}$$

Obviously, by omitting the term $C(\cdot)\lambda(1-\lambda)(u-v)^2$ in inequality (1), and the term $\frac{C(\cdot)}{4}(u-v)^2$ in inequality (2), we immediately get the definition of a convex, or Jensen–convex stochastic processes introduced by K. Nikodem in [7], and A. Skowroński in [10], respectively.

2 Jensen inequalities

In this section, we present counterparts of Jensen–type inequalities for strongly Jensen–convex stochastic processes and a counterpart of the integral Jensen inequality for strongly convex stochastic processes. Let us recall two technical lemmas. The first one is a special case of Lemma 2 proved in [3], and the second one was proved in [7]. The proof of the second lemma in deterministic case can be found in [4].

Lemma 2.1. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly Jensen–convex with modulus $C(\cdot)$ if and only if the stochastic process $Y : I \times \Omega \to \mathbb{R}$ defined by $Y(t,\cdot) := X(t,\cdot) - C(\cdot)t^2$ is Jensen–convex.

Lemma 2.2. Let I be an open interval. If $X : I \times \Omega \to \mathbb{R}$ is a Jensen–convex stochastic process, then for all $n \in \mathbb{N}$ and for all $t_1, ..., t_n \in I$ holds

$$X\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) \leqslant \frac{1}{n}\sum_{i=1}^{n} X(t_i,\cdot) \quad \text{(a.e.)}. \tag{3}$$

Now, we present a Jensen–type inequality for strongly Jensen–convex stochastic processes.

Theorem 2.3. Let I be an open interval. If $X : I \times \Omega \to \mathbb{R}$ is a strongly Jensen–convex with modulus $C(\cdot)$ stochastic process, then for all $n \in \mathbb{N}$ and for all $t_1, ..., t_n \in I$, we have

$$X\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) \leqslant \frac{1}{n}\sum_{i=1}^{n} X(t_i,\cdot) - \frac{C(\cdot)}{n}\sum_{i=1}^{n}\Big(t_i - \frac{1}{n}\sum_{i=1}^{n} t_i\Big)^2 \quad \text{(a.e.)}. \tag{4}$$

Proof. Fix $n \in \mathbb{N}$ and $t_1, ..., t_n \in I$. Since X is a strongly Jensen–convex stochastic process with modulus $C(\cdot)$, then by Lemma 2.1 there exists a Jensen–convex stochastic process $Y : I \times \Omega \to \mathbb{R}$, such that $X(t, \cdot) = Y(t, \cdot) + C(\cdot)t^2 \quad$ (a.e.). Process Y satisfies the inequality (3). It means

$$Y\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) \leqslant \frac{1}{n}\sum_{i=1}^{n} Y(t_i, \cdot) \quad \text{(a.e.)}.$$

Substituting in the above inequalities, the expression $Y(t, \cdot) = X(t, \cdot) - C(\cdot)t^2 \quad$ (a.e.), we have

$$X\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) - C(\cdot)\Big(\frac{1}{n}\sum_{i=1}^{n} t_i\Big)^2 \leqslant \frac{1}{n}\Big[\sum_{i=1}^{n} \{X(t_i, \cdot) - C(\cdot)t_i^2\}\Big] \quad \text{(a.e.)},$$

therefore

$$X\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) \leqslant \frac{1}{n}\sum_{i=1}^{n} X(t_i, \cdot) - C(\cdot)\underbrace{\Big[\frac{1}{n}\sum_{i=1}^{n} t_i^2 - \Big(\frac{1}{n}\sum_{i=1}^{n} t_i\Big)^2\Big]}_{=A} \quad \text{(a.e.)}.$$

To simplify the notation, we transform only the expression A. We put also $s := \frac{1}{n}\sum_{i=1}^{n} t_i$.

$$\begin{aligned} A = \frac{1}{n}\sum_{i=1}^{n} t_i^2 - \Big(\frac{1}{n}\sum_{i=1}^{n} t_i\Big)^2 = \frac{1}{n}\sum_{i=1}^{n} t_i^2 - \big(s\big)^2 = \frac{1}{n}\sum_{i=1}^{n}\big(t_i - s + s\big)^2 - \big(s\big)^2 = \\ \frac{1}{n}\sum_{i=1}^{n}\Big[\big(t_i - s\big)^2 + 2\big(t_i - s\big)s + \big(s\big)^2\Big] - \big(s\big)^2 = \\ \frac{1}{n}\sum_{i=1}^{n}\big(t_i - s\big)^2 + 2\frac{1}{n}s\Big[\sum_{i=1}^{n}\Big(t_i - s\Big)\Big] + \frac{1}{n}\sum_{i=1}^{n}\big(s\big)^2 - \big(s\big)^2 = \\ \frac{1}{n}\sum_{i=1}^{n}\big(t_i - s\big)^2 + 2\frac{1}{n}s\underbrace{\Big[\sum_{i=1}^{n} t_i - ns\Big]}_{=0} + \underbrace{\frac{1}{n}n\big(s\big)^2 - \big(s\big)^2}_{=0} = \frac{1}{n}\sum_{i=1}^{n}\big(t_i - s\big)^2. \end{aligned}$$

Finally

$$X\Big(\frac{1}{n}\sum_{i=1}^{n} t_i, \cdot\Big) \leqslant \frac{1}{n}\sum_{i=1}^{n} X(t_i, \cdot) - \frac{C(\cdot)}{n}\sum_{i=1}^{n}\Big(t_i - \frac{1}{n}\sum_{i=1}^{n} t_i\Big)^2 \quad \text{(a.e.)}.$$

□

Now, we extend the above theorem to convex combination with arbitrary rational coefficients.

Theorem 2.4. Let I be an open interval. If $X : I \times \Omega \to \mathbb{R}$ is a strongly Jensen–convex with modulus $C(\cdot)$ stochastic process, then the following inequality holds

$$X\Big(\sum_{i=1}^{n} q_i t_i, \cdot\Big) \leqslant \sum_{i=1}^{n} q_i X(t_i, \cdot) - C(\cdot) \sum_{i=1}^{n} q_i \Big(t_i - \sum_{i=1}^{n} q_i t_i\Big)^2 \quad \text{(a.e.)},$$

for all $t_1, ..., t_n \in I$ and $q_1, ..., q_n \in \mathbb{Q} \cap (0, 1)$, such that $q_1 + \cdots + q_n = 1$.

Proof. Fix $t_1, ..., t_n \in I$ and $q_1 = \frac{k_1}{l_1}, ..., q_n = \frac{k_n}{l_n} \in \mathbb{Q} \cap (0, 1)$ such that $q_1 + \cdots + q_n = 1$. Without loss of generality we may assume that $l_1 = \cdots = l_n = l$. Then $k_1 + \cdots + k_n = l$. We put $u_{11} = \cdots = u_{1k_1} =: t_1$, $u_{21} = \cdots = u_{2k_2} =: t_2$, ... , $u_{n1} = \cdots = u_{nk_n} =: t_n$, then

$$\sum_{i=1}^{n} q_i t_i = \frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} u_{ij}.$$

By Theorem 2.3, we get

$$X\Big(\sum_{i=1}^{n} q_i t_i, \cdot\Big) = X\Big(\frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} u_{ij}, \cdot\Big) \leqslant$$

$$\frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} X(u_{ij}, \cdot) - \frac{C(\cdot)}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} \Big(u_{ij} - \frac{1}{l} \sum_{i=1}^{n} \sum_{j=1}^{k_i} u_{ij}\Big)^2 =$$

$$\sum_{i=1}^{n} q_i X(t_i, \cdot) - C(\cdot) \sum_{i=1}^{n} q_i \Big(t_i - \sum_{i=1}^{n} q_i t_i\Big)^2 \quad \text{(a.e.)}.$$

□

By the above theorem, we obtain the following corollary.

Corollary 2.5. Let I be an open interval. A stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly Jensen–convex with modulus $C(\cdot)$ if and only if

$$X\big(qu + (1-q)v, \cdot\big) \leqslant qX(u, \cdot) + (1-q)X(v, \cdot) - C(\cdot)q(1-q)\big(u - v\big)^2 \quad \text{(a.e.)},$$

for every $u, v \in I$ and $q \in \mathbb{Q} \cap (0, 1)$.

Now, we prove a counterpart of the integral Jensen inequality for strongly convex stochastic processes.

Let $([a, b], \Lambda, \mu)$ be a probability measure space, where $[a, b] \subset \mathbb{R}$, Λ is the sigma algebra of Lebesgue measurable sets, $\mu = \frac{1}{b-a}\lambda$ is a probability measure ($\mu\big([a, b]\big) = 1$). We denote by λ the Lebesgue measure.

Theorem 2.6. Let $X : I \times \Omega \to \mathbb{R}$ be a strongly convex with modulus $C(\cdot)$ stochastic process, $\varphi : [a,b] \to I$ be a square integrable function according to the measure μ. Then

$$X\big(m,\cdot\big) \leqslant \int_a^b X\big(\varphi(t),\cdot\big)d\mu - C(\cdot)\int_a^b \big(\varphi(t)-m\big)^2 d\mu \quad \text{(a.e.)},$$

where $m = \int_a^b \varphi(t)d\mu$.

Proof. Since X is strongly convex, then it is convex too. By Corollary 3 proved in [3], there exists a stochastic process $H(t;\cdot) = C(\cdot)(t-t_0)^2 + A(\cdot)(t-t_0) + X(t_0,\cdot)$ supporting X in $t_0 \in \operatorname{int} I$. It means, for every $t_0 \in \operatorname{int} I$, the following inequality holds

$$X(\varphi(t),\cdot) \geqslant C(\cdot)(\varphi(t)-t_0)^2 + A(\cdot)(\varphi(t)-t_0) + X(t_0,\cdot) \quad \text{(a.e.)}, \tag{5}$$

where $A : \Omega \to \mathbb{R}$ is a random variable. By the mean–value theorem $m = \int_a^b \varphi(t)d\mu \in I$. We take the inequality (5) for m and we get

$$X(\varphi(t),\cdot) \geqslant C(\cdot)(\varphi(t)-m)^2 + A(\cdot)(\varphi(t)-m) + X(m,\cdot) \quad \text{(a.e.)}.$$

Integrating the above inequality according to the measure μ, we get

$$\begin{aligned}
&\int_a^b X(\varphi(t),\cdot)d\mu \geqslant \\
&\qquad C(\cdot)\int_a^b (\varphi(t)-m)^2 d\mu + A(\cdot)\int_a^b (\varphi(t)-m)d\mu + X(m,\cdot)\int_a^b d\mu = \\
&C(\cdot)\int_a^b (\varphi(t)-m)^2 d\mu + A(\cdot)\Big[\int_a^b \varphi(t)d\mu - m\int_a^b d\mu\Big] + X(m,\cdot)\int_a^b d\mu = \\
&C(\cdot)\int_a^b (\varphi(t)-m)^2 d\mu + A(\cdot)\Big[m - m\mu\big([a,b]\big)\Big] + X(m,\cdot)\mu\big([a,b]\big) \quad \text{(a.e.)}.
\end{aligned}$$

By the probability of the measure μ, we have

$$\int_a^b X(\varphi(t),\cdot)d\mu \geqslant C(\cdot)\int_a^b (\varphi(t)-m)^2 d\mu + X(m,\cdot) \quad \text{(a.e.)},$$

which completes the proof. □

3 Fejer and Hermite–Hadamard inequalities

In this section, we present counterparts of well known Fejer and Hermite–Hadamard inequalities for strongly convex stochastic processes. In deterministic case these facts were described in [1] and [6].

Let us recall before that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is *mean-square continuous* in the interval $[a, b]$, if for all $t_0 \in I$ the condition

$$\lim_{t \to t_0} E(|X(t) - X(t_0)|^2) = 0$$

holds. In this section, we use the notion of *mean–square integral*. For the definition and basic properties of mean-square integral see for example [11]. We start our investigation with the following technical lemma. It can be easily prove by basic mean–square integral properties, so we omit the proof.

Lemma 3.1. Let $G : I \times \Omega \to \mathbb{R}_+$ be a mean–square integrable stochastic process, such that $G(a + b - t) = G(t)$ (a.e.) for all $t \in [a, b] \subset I$, and

$$\int_a^b G(t, \cdot)dt = J(\cdot) \quad \text{(a.e.)},$$

where $J : \Omega \to \mathbb{R}$ is a unit random variable. Then

$$\int_a^b tG(t, \cdot)dt = \frac{a + b}{2} J(\cdot) \quad \text{(a.e.)}. \tag{6}$$

The following theorem is a counterpart of Fejer inequality for strongly convex stochastic processes.

Theorem 3.2. Let $X : [a, b] \times \Omega \to \mathbb{R}$ be a strongly convex with modulus $C(\cdot)$, mean-square continuous in $[a, b]$ stochastic process. Let $G : [a, b] \times \Omega \to \mathbb{R}_+$ be a *mean–square integrable* stochastic process, such that $G(a+b-t, \cdot) = G(t, \cdot)$ (a.e.) for all $t \in [a, b]$, and

$$\int_a^b G(t, \cdot)dt = J(\cdot) \quad \text{(a.e.)},$$

where $J : \Omega \to \mathbb{R}$ is a unit random variable. The following inequality holds

$$X\Big(\frac{a + b}{2}, \cdot\Big) + C(\cdot)\Big[\int_a^b t^2 G(t, \cdot)dt - \Big(\frac{a + b}{2}\Big)^2\Big] \leqslant \int_a^b X(t, \cdot)G(t, \cdot)dt \leqslant \frac{X(a, \cdot) + X(b, \cdot)}{2} - C(\cdot)\Big[\frac{a^2 + b^2}{2} - \int_a^b t^2 G(t, \cdot)dt\Big] \quad \text{(a.e.)}. \tag{7}$$

Proof. To prove the left-hand side of (7), we take $s = \frac{a+b}{2}$ and a process of the form $H(t, \cdot) = C(\cdot)(t - s)^2 + A(\cdot)(t - s) + X(s, \cdot)$ supporting X in s (see Corollary 3 [3]). By the basic properties of mean–square integral, we have

$$\int_a^b X(t, \cdot)G(t, \cdot)dt \geqslant \int_a^b H(t, \cdot)G(t, \cdot)dt = C(\cdot)\int_a^b t^2 G(t, \cdot)dt + \big(-2C(\cdot)s + A(\cdot)\big)\int_a^b tG(t, \cdot)dt + \big(C(\cdot)s^2 - A(\cdot)s + X(s, \cdot)\big)\int_a^b G(t, \cdot)dt \quad \text{(a.e.)}.$$

By Lemma 3.1, basic properties of mean–square integral and the assumption about G, we get

$$\int_a^b X(t,\cdot)G(t,\cdot)dt \geqslant C(\cdot)\int_a^b t^2G(t,\cdot)dt - C(\cdot)s^2 + X(s,\cdot) = X\Big(\frac{a+b}{2},\cdot\Big) + C(\cdot)\Big[\int_a^b t^2G(t,\cdot)dt - \Big(\frac{a+b}{2}\Big)^2\Big] \quad \text{(a.e.)}.$$

To prove the right-hand side of (7), we use inequality (1) for the following convex combination $t = \frac{b-t}{b-a}a + \frac{t-a}{b-a}b$. By strongly convexity of X and basic properties of mean–square integral we have

$$\int_a^b X(t,\cdot)G(t,\cdot)dt = \int_a^b X\Big(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b,\cdot\Big)G(t,\cdot)dt \leqslant \int_a^b \Big[\frac{b-t}{b-a}X(a,\cdot) + \frac{t-a}{b-a}X(b,\cdot) - C(\cdot)\frac{(b-x)(x-a)}{(b-a)^2}(b-a)^2\Big]G(t,\cdot)dt = \int_a^b \Big[\frac{bX(a,\cdot)-aX(b,\cdot)}{b-a} + \frac{X(b,\cdot)-X(a,\cdot)}{b-a}t - C(\cdot)\big((a+b)t-ab-t^2\big)\Big]G(t,\cdot)dt \quad \text{(a.e.)}.$$

Finally

$$\int_a^b X(t,\cdot)G(t,\cdot)dt \leqslant \frac{bX(a,\cdot)-aX(b,\cdot)}{b-a} + \frac{X(b,\cdot)-X(a,\cdot)}{b-a}\cdot\frac{a+b}{2} - C(\cdot)\Big[\frac{(a+b)^2}{2} - ab - \int_a^b t^2G(t,\cdot)dt\Big] = \frac{X(a,\cdot)+X(b,\cdot)}{2} - C(\cdot)\Big[\frac{a^2+b^2}{2} - \int_a^b t^2G(t,\cdot)dt\Big] \quad \text{(a.e.)}.$$

□

Note that, if we put $C(\cdot) = 0$ in (7), then we get Fejer inequality for convex stochastic processes.

$$X(\frac{a+b}{2},\cdot) \leqslant \int_a^b X(t,\cdot)G(t,\cdot)dt \leqslant \frac{X(a,\cdot)+X(b,\cdot)}{2} \quad \text{(a.e.)}. \tag{8}$$

Using the inequality (8) for the process $X(t,\cdot) = t^2J(\cdot)$ we have

$$\Big(\frac{a+b}{2}\Big)^2 \leqslant \int_a^b t^2G(t,\cdot)dt \leqslant \frac{a^2+b^2}{2} \quad \text{(a.e.)}.$$

By the above inequality, the terms

$$\int_a^b t^2G(t,\cdot)dt - \Big(\frac{a+b}{2}\Big)^2 \quad \text{and} \quad \frac{a^2+b^2}{2} - \int_a^b t^2G(t,\cdot)dt$$

in the inequality (7) are nonnegative. In consequence, the inequality (7) is stronger then the inequality (8). Note also, that Fejer inequality (7) generalizes Hermite–Hadamard inequality proved in [3]. Indeed, for $G(t,\cdot) = \frac{1}{b-a}$ the inequality (7), can be written in the form

$$X\Big(\frac{u+v}{2},\cdot\Big) + C(\cdot)\frac{(v-u)^2}{12} \leqslant \frac{1}{v-u}\int_u^v X(t,\cdot)\,\mathrm{d}t \leqslant \frac{X(u,\cdot)+X(v,\cdot)}{2} - C(\cdot)\frac{(u-v)^2}{6} \quad \text{(a.e.).} \tag{9}$$

The next result shows that the converse of Hermite–Hadamard theorem for strongly convex stochastic processes, is also valid.

Theorem 3.3. Let a stochastic process $X : I \times \Omega \to \mathbb{R}$ be mean–square continuous in the interval I, and let it satisfy the left or right hand side inequality in (9). Then X is strongly convex.

Proof. First we will prove the theorem in the case when the left hand side inequality of (9) holds. Let us define a stochastic process $Y : I \times \Omega \to \mathbb{R}$, such that $Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$, where $C : \Omega \to \mathbb{R}$ is a random variable occurring in (9). Substituting the expression $X(t,\cdot) = Y(t,\cdot)+C(\cdot)t^2$ to the left hand side inequality of (9) we get

$$Y\Big(\frac{u+v}{2},\cdot\Big) + C(\cdot)\Big(\frac{u+v}{2}\Big)^2 + C(\cdot)\frac{(v-u)^2}{12} \leqslant \frac{1}{v-u}\int_u^v \Big(Y(t,\cdot)+C(\cdot)t^2\Big)\,\mathrm{d}t \quad \text{(a.e.).} \tag{10}$$

By the basic properties of mean–square integral we have

$$Y\Big(\frac{u+v}{2},\cdot\Big) + C(\cdot)\frac{4u^2+4uv+4v^2}{12} \leqslant \frac{1}{v-u}\int_u^v Y(t,\cdot)\,\mathrm{d}t + C(\cdot)\frac{1}{v-u}\frac{v^3-u^3}{3} \quad \text{(a.e.).} \tag{11}$$

Subtracting by sides in (11) the term $C(\cdot)\frac{u^2+uv+v^2}{3}$ we get

$$Y\Big(\frac{u+v}{2},\cdot\Big) \leqslant \frac{1}{v-u}\int_u^v Y(t,\cdot)\,\mathrm{d}t \quad \text{(a.e.).}$$

This means that Y satisfy the left hand side inequality of the Hermite–Hadamard inequality for convex stochastic processes. By Theorem 6 [2] Y is convex. By Lemma 2 [3] the stochastic process X is strongly convex with modulus $C(\cdot)$.

Now, let the right hand side of the inequality (9) be satisfied. As before, we define a stochastic process $Y : I \times \Omega \to \mathbb{R}$, such that $Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$, where $C : \Omega \to \mathbb{R}$ is a random variable occurring in (9). Substituting the expression $X(t,\cdot) = Y(t,\cdot) + C(\cdot)t^2$ to the right hand side inequality of (9) we get

$$\frac{1}{v-u}\int_u^v \Big(Y(t,\cdot)+C(\cdot)t^2\Big)\,\mathrm{d}t \leqslant \frac{Y(u,\cdot)+Y(v,\cdot)}{2} + C(\cdot)\frac{u^2+v^2}{2} - C(\cdot)\frac{(u-v)^2}{6} \quad \text{(a.e.).} \tag{12}$$

By the basic properties of mean–square integral we have

$$\frac{1}{v-u}\int_u^v Y(t,\cdot)\,\mathrm{d}t + C(\cdot)\frac{1}{v-u}\frac{v^3-u^3}{3} \leqslant \frac{Y(u,\cdot)+Y(v,\cdot)}{2} + C(\cdot)\frac{2u^2+2uv+2v^2}{6} \quad \text{(a.e.).} \tag{13}$$

Subtracting by sides in (13) the term $C(\cdot)\frac{u^2+uv+v^2}{3}$ we get

$$\frac{1}{v-u}\int_u^v Y(t,\cdot)\,\mathrm{d}t \leqslant \frac{Y(u,\cdot)+Y(v,\cdot)}{2} \quad \text{(a.e.).}$$

Thus Y satisfy the right hand side inequality of the Hermite–Hadamard inequality for convex stochastic processes. By Theorem 6 [2] Y is convex. By Lemma 2 [3] the stochastic process X is strongly convex with modulus $C(\cdot)$. □

References

[1] Azócar A., Giménez J., Nikodem K., Sánchez J. L.: *On strongly midconvex functions*, Opuscula Math. 31/1 (2011), 15–26.

[2] Kotrys D.: *Hermite–Hadamard inequality for convex stochastic processes*, Aequationes Math. 83 , 143–151 (2012).

[3] Kotrys, D.: *Remarks on strongly convex stochastic processes*, Aequat. Math. 86, 91–98 (2012).

[4] Kuczma M.: *An Introduction to the Theory of Functional Equations and Inequalities*, Birkhäuser, Basel 2009.

[5] Lara T., Merentes N., Rosales E., Valera M.: *Some characterizations of strongly convex functions in inner product spaces*, Mathematica Aeterna Vol. 4 , no. 6, 651–657 (2014).

[6] Merentes N., Nikodem K.: *Remarks on strongly convex functions*, Aequat. Math. 80, 193–199 (2010).

[7] Nikodem, K.: *On convex stochastic processes*, Aequat. Math. 20, 184–197 (1980).

[8] Polyak B.T.: *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl. 7 , 72–75 (1966).

[9] Roberts A. W., Varberg D. E.: *Convex functions*, Academic Press, New York–London (1973).

[10] Skowroński, A.: *On some properties of J-convex stochastic processes*, Aequat. Math. 44, 249–258 (1992).

[11] Sobczyk K.: *Stochastic differential equations with applications to physics and engineering*, Kluwer Academic Publishers B.V. (1991).

Received: January, 2015