Relation between S-Metric And $\mathcal{M}$-Fuzzy Metric Spaces

Zeinab Hassanzadeh
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
Z.hassanzadeh1368@yahoo.com

Atena Javaheri
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
Javaheri.a91@gmail.com

Shaban Sedghi
Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
sedghi.gh@yahoo.com

Nabi Shobe
Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran
nabi_shobe@yahoo.com

Abstract
In this work we have considered several common fixed point results in S-metric spaces for weak compatible mappings. By applications of these results we establish some fixed point theorems in fuzzy S-metric spaces.

Mathematics Subject Classification: 54E40; 54E35; 54H25

Keywords: Fuzzy contractive mapping; Complete fuzzy metric space; Common fixed point theorem; R-weakly commuting maps.

1The corresponding author: sedghi.gh@yahoo.com (Shaban Sedghi Ghadikolaei).

1 Introduction
In this paper we establish some fixed point results in a fuzzy S-metric space by applications of certain fixed point theorems in S-metric spaces. Also we
prove some fixed point results in S-metric spaces. Fuzzy metric space was first introduced by Kramosil and Michalek [13]. Subsequently, George and Veeramani had given a modified definition of fuzzy metric spaces [5]. Fixed point results in such spaces have been established in a large number of works. Some of these works are noted in [6, 17, 15, 23, 24, 25].

**Definition 1.1** [5] A binary operation * : [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions:

1. * is associative and commutative,
2. * is continuous,
3. a * 1 = a for all a ∈ [0, 1],
4. a * b ≤ c * d whenever a ≤ c and b ≤ d, for each a, b, c, d ∈ [0, 1].

Two typical examples of continuous t-norm are a * b = ab and a * b = \min(a, b).

Here we have considered definition of fuzzy metric space (non-Archimedean).

**Definition 1.2** [16] A 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t–norm and M is a fuzzy set on \(X^2 \times (0, \infty)\) satisfying the following conditions for each \(x, y, z \in X\) and \(t, s > 0\):

1. \(M(x, y, t) > 0\),
2. \(M(x, y, t) = 1\) if and only if \(x = y\),
3. \(M(x, y, t) = M(y, x, t)\),
4. \(M(x, z, t) * M(z, y, s) \leq M(x, y, t \vee s)\), where \(t \vee s = \max\{s, t\}\),
5. \(M(x, y, .) : (0, \infty) \rightarrow [0, 1]\) is continuous.

All fuzzy metric in this paper are assumed to be non-Archimedean.

In 1976, Jungck [8] introduced the notion of commuting mappings to find common fixed point results in metric spaces. Later on, in [9] Jungck proposed the notion of compatible mappings which is a generalization of the concept of commuting mapping. Some common fixed point theorems for compatible mappings and their generalizations are addressed in [10, 11, 14, 26]. In this paper we consider weak compatible mappings.

**Definition 1.3** [18] Let A and S be mappings from a metric space X into itself. Then the mappings are said to be weak compatible if they commute at a coincidence point, that is, \(Ax = Sx\) implies that \(ASx = SAx\).

## 2 Preliminary Notes

First we recall some notions, lemmas, and examples which will be useful later.
Definition 2.1 [21] Let $X$ be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be an $S$-metric on $X$, if for each $x, y, z, a \in X$,

1. $S(x, y, z) \geq 0$,
2. $S(x, y, z) = 0$ if and only if $x = y = z$,
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair $(X, S)$ is called an $S$-metric space.

Example 2.2 [21] We can easily check that the following examples are $S$-metric spaces.

1. Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on $X$. Then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an $S$-metric on $X$.

In general, if $X$ is a vector space over $\mathbb{R}$ and $\| \cdot \|$ a norm on $X$. Then it is easy to see that

\[ S(x, y, z) = ||\alpha y + \beta z - \lambda x|| + ||y - z||, \]

where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an $S$-metric on $X$.

2. Let $X$ be a nonempty set and $d_1$, $d_2$ be two ordinary metrics on $X$. Then

\[ S(x, y, z) = d_1(x, z) + d_2(y, z), \]

is an $S$-metric on $X$.

Lemma 2.3 [19] Let $(X, S)$ be an $S$-metric space. Then, we have $S(x, x, y) = S(y, y, x)$, $x, y \in X$.

For more detail of $S$-metric see the reference [20].

Definition 2.4 [20] Let $(X, S)$ be an $S$-metric space and $A \subset X$.

1. A sequence $\{x_n\}$ in $X$ converges to $x$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$. This case, we denote by $\lim_{n\to \infty} x_n = x$ and we say that $x$ is the limit of $\{x_n\}$ in $X$.

2. A sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

3. The $S$-metric space $(X, S)$ is said to be complete if every Cauchy sequence is convergent.

Lemma 2.5 [20] Let $(X, S)$ be an $S$-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to \infty} x_n = x$ and $\lim_{n\to \infty} y_n = y$, then

\[ \lim_{n\to \infty} S(x_n, x_n, y_n) = S(x, y). \]
3 Main Results

Next we establish the following result in S-metric spaces.

**Theorem 3.1** Let $A, B, C$ and $T$ be self maps on a complete S-metric space $(X, S)$ satisfying:

(i) $A(X) \subseteq T(X), B(X) \subseteq C(X)$ and $T(X)$ or $C(X)$ is a closed subset of $X$;

(ii) there exist positive real numbers $a, b, c, e$ such that $a + b + c + 3e < 1$ and for each $x, y, z \in X$,

\[
S(Ax, Ay, Bz) \leq aS(Cx, Cx, Tz) + bS(Cx, Cx, Ax) + cS(Tz, Tz, Bz) + e(S(Cx, Cx, Bz) + S(Tz, Tz, Ay));
\]

(iii) the pairs $(A, C)$ and $(B, T)$ are weakly compatible.

Then $A, B, C$ and $T$ have a unique common fixed point in $X$.

**Proof** Let $x_0$ be an arbitrary point in $X$. By (i), we can choose a point $x_1$ in $X$ such that $y_0 = Ax_0 = Tx_1$ and $y_1 = Bx_1 = Cx_2$. In general, there exists a sequence $\{y_n\}$ such that, $y_{2n} = Ax_{2n} = T_{2n+1}$ and $y_{2n+1} = B_{2n+1} = C_{2n+2}$, for $n = 0, 1, 2, \ldots$. We claim that the sequence $\{y_n\}$ is a Cauchy sequence.

By (ii), we have,

\[
S(y_{2n}, y_{2n}, y_{2n+1}) = S(Ax_{2n}, Ax_{2n}, Bx_{2n+1}) \\
\leq aS(Cx_{2n}, Cx_{2n}, Tx_{2n+1}) + bS(Cx_{2n}, Cx_{2n}, Ax_{2n}) + \\
+ cS(Tx_{2n+1}, Tx_{2n+1}, Bx_{2n+1}) + e(S(Cx_{2n}, Cx_{2n}, Bx_{2n+1}) + S(Tx_{2n+1}, Tx_{2n+1}, Ax_{2n})) \\
= aS(y_{2n-1}, y_{2n-1}, y_{2n}) + bS(y_{2n-1}, y_{2n-1}, y_{2n}) + \\
+ cS(y_{2n}, y_{2n}, y_{2n+1}) + e(S(y_{2n-1}, y_{2n-1}, y_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n})).
\]

If we put $d_n = S(y_n, y_n, y_{n+1})$, then by above inequality we have,

\[
d_{2n} \leq ad_{2n-1} + bd_{2n-1} + cd_{2n} + e(S(y_{2n-1}, y_{2n-1}, y_{2n+1}) + 0) \\
\leq ad_{2n-1} + bd_{2n-1} + cd_{2n} + e(2S(y_{2n-1}, y_{2n-1}, y_{2n}) + S(y_{2n}, y_{2n}, y_{2n+1})).
\]

Hence,

\[
d_{2n} \leq ad_{2n-1} + bd_{2n-1} + cd_{2n} + 2ed_{2n-1} + ed_{2n}.
\]

Hence we have,

\[
d_{2n} \leq \frac{a + b + 2e}{1 - c - e}d_{2n-1} = td_{2n-1},
\]

where $t = \frac{a + b + 2e}{1 - c - e}$.
where $0 < t = \frac{a + b + 2e}{1 - e - b} < 1$.

Similarly, it follows that

$$d_{2n+1} \leq \frac{a + c + 2e}{1 - e - b} d_{2n} = td_{2n},$$

where $0 < t' = \frac{a + c + 2e}{1 - e - b} < 1$. If we set $k = \max\{t, t'\} < 1$, then for every $n \in \mathbb{N}$ by above inequalities we get $d_n \leq kd_{n-1}$.

Hence,

$$d_n \leq kd_{n-1} \leq k^2 d_{n-2} \leq \cdots \leq k^n d_0.$$  \hspace{1cm} (2)

That is,

$$S(y_n, y_n, y_{n+1}) \leq k^n S(y_0, y_0, y_1)$$  \hspace{1cm} (3)

If $m \geq n$, then

$$S(y_n, y_n, y_m) \leq 2S(y_n, y_n, y_{n+1}) + 2S(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots + 2S(y_{m-1}, y_{m-1}, y_m) \leq 2k^n S(y_0, y_0, y_1) + 2k^{n+1} S(y_0, y_0, y_1) + \cdots + 2k^{m-1} S(y_0, y_0, y_1) \leq \frac{2k^n}{1 - k} S(y_0, y_0, y_1) \to 0$$

as $n \to \infty$. It follows that, the sequence $\{y_n\}$ is Cauchy sequence and by the completeness of $X$, $\{y_n\}$ converges to $y \in X$. Then

$$\lim_{n \to \infty} y_n = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Cx_{2n+2} = \lim_{n \to \infty} Tx_{2n+1} = y. \hspace{1cm} (4)$$

Let $T(X)$ be a closed subset of $X$, then there exists $v \in X$ such that $Tv = y$.

We now prove that $Bv = y$. By (ii), we get

$$\lim_{n \to \infty} S(Ax_{2n}, Ax_{2n}, Bv) \leq \lim_{n \to \infty} [aS(Cx_{2n}, Cx_{2n}, Tv) + bS(Ax_{2n}, Ax_{2n}, Cx_{2n}) + cS(Bv, Bv, Tv) + e(S(Bv, Bv, Cx_{2n}) + S(Ax_{2n}, Ax_{2n}, Tv))]$$

and so

$$S(y, y, Bv) \leq aS(y, y, Tv) + bS(y, y, y) + cS(Bv, Bv, y) + e(S(Bv, Bv, y) + S(y, y, Tv))< S(y, y, Bv).$$

It follows that $Bv = y = Tv$. Since $B$ and $T$ are two weakly compatible mappings, we have $BTv = TBv$ and so $By = Ty$.

Next, we prove that $By = y$. By (ii), we get

$$\lim_{n \to \infty} S(Ax_{2n}, Ax_{2n}, By) \leq \lim_{n \to \infty} [aS(Cx_{2n}, Cx_{2n}, Ty) + bS(Ax_{2n}, Ax_{2n}, Cx_{2n}) + cS(By, By, Ty) + e(S(By, By, Cx_{2n}) + S(Ax_{2n}, Ax_{2n}, Ty))].$$
Hence,
\[ S(y, y, By) \leq aS(y, y, Ty) + bS(y, y, y) + cS(By, By, Ty) + e(S(By, By, y) + S(y, y, Ty)) < S(y, y, By) \]
and so \( By = y \).

Since \( B(X) \subseteq C(X) \), there exists \( w \in X \) such that \( Cw = y \). We prove that
\[ S(Aw, Aw, By) \leq aS(Cw, Cw, Ty) + bS(Aw, Aw, Cw) + cS(By, By, Ty) + e(S(By, By, Cw) + S(Aw, Aw, Ty)) \]
and it follows that
\[ S(Aw, Aw, y) \leq aS(y, y, y) + bS(Aw, Aw, y) + cS(y, y, Ty) + e(S(y, y, y) + S(Aw, Aw, Ty)) < S(Aw, Aw, y). \]

This implies that \( Aw = y \) and hence \( Aw = Cw = y \). Since \( A \) and \( C \) are weakly compatible, then \( ACw = CAw \) and so \( Ay = Cy \).

Now, we prove that \( Ay = y \). From (ii), we have
\[ S(Ay, Ay, By) \leq aS(Cy, Cy, Ty) + bS(Ay, Ay, Cy) + cS(By, By, Ty) + e(S(By, By, Cy) + S(Ay, Ay, Ty)) \]
and hence \( Ay = y \) and therefore \( Ay = Cy = By = Ty = y \). That is \( y \) is a common fixed point for \( A, B, C, T \).

The proof is similar when \( C(X) \) is assumed to be a closed subset of \( X \).

Now to prove the uniqueness. Assume that \( x \) is another common fixed point of \( A, B, C \) and \( T \). Then
\[ S(x, x, y) = S(Ax, Ax, By) \leq aS(Cx, Cx, Ty) + bS(Ax, Ax, Cx) + cS(By, By, Ty) + e(S(Cx, Cx, By) + S(Ax, Ax, Ty)) \]
and so
\[ S(x, x, y) \leq aS(x, x, y) + bS(x, x, x) + cS(y, y, y) + e(S(x, x, y) + S(x, x, y)) < S(x, x, y). \]
Thus it follows that \( x = y \).
Corollary 3.2 Let \( A \) and \( B \) be self maps on a complete S-metric space \((X, S)\) satisfying: there exist positive real numbers \( a, b, c, e \) such that \( a + b + c + 3e < 1 \) and for each \( x, y, z \in X \),
\[
S(Ax, Ay, Bz) \leq aS(x, x, z) + bS(x, x, Ax) + cS(z, z, Bz) + e(S(x, x, Bz) + S(z, z, Ay));
\]

Then \( A \) and \( B \) have a unique common fixed point in \( X \).

Proof 2 It is enough set \( T = C = I \), identity map in Theorem 3.1.

Here we introduce \( M \)-fuzzy metric. We describe the space along with some associated concepts in the following.

Definition 3.3 A 3-tuple \((X, M, \ast)\) is called a \( M \)-fuzzy metric space if \( X \) is an arbitrary (non-empty) set, \( \ast \) is a continuous \( t \)-norm and \( M \) is a fuzzy set on \( X^3 \times (0, \infty) \) satisfying the following conditions for each \( x, y, z, a \in X \) and \( t, s, r > 0 \):

1. \( M(x, y, z, t) > 0 \),
2. \( M(x, y, z, t) = 1 \) if and only if \( x = y = z \),
3. \( M(x, y, z, \vee\{t, s, r\}) \geq M(x, x, a, t) \ast M(y, y, a, s) \ast M(z, z, a, r) \) where \( \vee\{t, s, r\} = \max\{t, s, r\} \),
4. \( M(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1] \) is continuous.

Example 3.4 Let \( a \ast b = ab \) for all \( a, b \in [0, 1] \), we define
\[
M(x, y, z, t) = \exp\left(\frac{S(x, y, z)}{t}\right),
\]
where \( S \) is an S-metric on set \( X \). Then \((X, M, \ast)\) is a \( M \)-fuzzy metric space.

Proof 3 (i) \( M(x, y, z, t) > 0 \) for all \( x, y, z \in X \) and \( t > 0 \) is trivial.

(ii) \( M(x, y, z, t) = 1 \iff S(x, y, z) = 0 \iff x = y = z \).

(iii) Since \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \), hence,
\[
\frac{S(x, y, z)}{t \vee s \vee r} \leq \frac{S(x, x, a)}{t} + \frac{S(y, y, a)}{s} + \frac{S(z, z, a)}{r}.
\]
Thus

\[
(iii) \quad \exp \frac{-S(x, y, z)}{t \vee s \vee r} \geq \exp \left( -\frac{S(x, x, a)}{t^2} - \frac{S(y, y, a)}{s^2} - \frac{S(z, z, a)}{r^2} \right) \\
= \exp \frac{-S(x, y, z)}{t} \exp \frac{-S(x, x, a)}{s} \exp \frac{-S(z, z, a)}{r},
\]

it follows that,

\[
(iii) \mathcal{M}(x, y, z, t \vee s \vee r) \geq \mathcal{M}(x, x, a, t) \mathcal{M}(y, y, a, s) \mathcal{M}(z, z, a, r) \\
= \mathcal{M}(x, x, a, t) \ast \mathcal{M}(y, y, a, s) \ast \mathcal{M}(z, z, a, r).
\]

\((X, \mathcal{M}, \ast)\) is a \(\mathcal{M}\)-fuzzy metric space.

A sequence \(\{x_n\}\) in \(X\) converges to \(x\) if and only if \(\mathcal{M}(x_n, x_n, x, t) \rightarrow 1\) as \(n \rightarrow \infty\), for each \(t > 0\). It is called a Cauchy sequence if for each \(0 < \varepsilon < 1\) and \(t > 0\), there exist \(n_0 \in \mathbb{N}\) such that \(\mathcal{M}(x_n, x_n, x_m, t) > 1 - \varepsilon\) for each \(n, m \geq n_0\). The \(\mathcal{M}\)-fuzzy metric space \((X, \mathcal{M}, \ast)\) is said to be complete if every Cauchy sequence is convergent.

The following properties of \(\mathcal{M}\) noted in the theorem below are easy consequences of the definition.

**Lemma 3.5** Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space. Then

1. \(\mathcal{M}(x, x, y, t) = \mathcal{M}(y, y, x, t)\).
2. \(\mathcal{M}(x, x, y, t)\) is nondecreasing with respect to \(t\) for each \(x, y \in X\).

**Proof 4** (i) For every \(t \in (0, \infty)\), we have

\[
\mathcal{M}(x, x, y, t) = \mathcal{M}(x, x, y, t \vee t \vee t) \geq \mathcal{M}(x, x, x, t) \ast \mathcal{M}(x, x, x, t) \ast \mathcal{M}(y, y, x, t) \\
= \mathcal{M}(y, y, x, t).
\]

Similarly, we can show that \(\mathcal{M}(y, y, x, t) \geq \mathcal{M}(x, x, y, t)\). That is \(\mathcal{M}(x, x, y, t) = \mathcal{M}(y, y, x, t)\).

(ii) For every \(t, s \in (0, \infty)\), let \(t \geq s\). Then

\[
\mathcal{M}(x, x, y, t) = \mathcal{M}(x, x, y, t \vee s \vee s) \geq \mathcal{M}(x, x, x, t) \ast \mathcal{M}(x, x, x, s) \ast \mathcal{M}(y, y, x, s) \\
= \mathcal{M}(y, y, x, s) = \mathcal{M}(x, x, y, s).
\]

**Example 3.6** Let \(a \ast b = ab\) for all \(a, b \in [0, 1]\) and \(M_1\) and \(M_2\) be two fuzzy set on \(X \times X \times (0, +\infty)\) defined by

\[
\mathcal{M}(x, y, z, t) = M_1(x, z, t) \ast M_2(y, z, t),
\]

for all \(x, y, z \in X\). Then \((X, \mathcal{M}, \ast)\) is a \(\mathcal{M}\)-fuzzy metric space.
Proof 5 (i) \( \mathcal{M}(x, y, z, t) > 0 \) for all \( x, y, z \in X \) and \( t > 0 \) is trivial.

\[ (ii) \mathcal{M}(x, y, z, t) = 1 \iff M_1(x, z, t) = M_2(y, z, t) = 1 \iff x = y = z. \]

(iii) Let \( t \geq s \geq r \), it follows that,
\[
\mathcal{M}(x, y, z, t \vee s \vee r) = M_1(x, y, z, t) \geq M_1(x, a, t) \ast M_1(a, z, t) \ast M_2(y, a, t) \ast M_2(z, a, t)
\]
\[ \geq M_1(x, a, t) \ast M_2(x, a, t) \ast M_1(y, a, t) \ast M_2(y, a, t) \ast M_1(z, a, t) \ast M_2(z, a, t) = \mathcal{M}(x, y, a, t) \ast \mathcal{M}(y, y, a, t) \ast \mathcal{M}(z, z, a, t) \]
\[ \geq \mathcal{M}(x, x, a, t) \ast \mathcal{M}(y, y, a, t) \ast \mathcal{M}(z, z, a, t), \]

\((X, \mathcal{M}, \ast)\) is a \( \mathcal{M}\)-fuzzy metric space.

Lemma 3.7 Let \((X, \mathcal{M}, \ast)\) be a \( \mathcal{M}\)-fuzzy metric space. If sequence \( \{x_n\}\) in \( X \) converges to \( x \), then \( x \) is unique.

Proof 6 Let \( \{x_n\} \) converges to \( x \) and \( y \), then for each \( 0 < \varepsilon < 1 \) there exist \( n_1, n_2 \in N \) such that
\[ \forall \ n \geq n_1 \implies \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon, \tag{7} \]
and
\[ \forall \ n \geq n_2 \implies \mathcal{M}(x_n, x_n, y, t) > 1 - \varepsilon. \tag{8} \]
If set \( n_0 = \max\{n_1, n_2\} \), then for every \( n \geq n_0 \) we have:
\[
\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, x, x, t) \ast \mathcal{M}(x, x, x, t) \ast \mathcal{M}(y, y, x, t) \ast \mathcal{M}(y, y, x, t) \ast \mathcal{M}(y, y, x, t) > (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon)
\]
By taking the limit when \( \varepsilon \to 0 \) in above inequality we get \( \mathcal{M}(x, x, y, t) \geq 1 \). Hence \( \mathcal{M}(x, x, y, t) = 1 \) so \( x = y \).

Lemma 3.8 Let \((X, \mathcal{M}, \ast)\) be a \( \mathcal{M}\)-fuzzy metric space. Then the convergent sequence \( \{x_n\} \) in \( X \) is Cauchy.

Proof 7 Since \( \lim_{n \to \infty} x_n = x \) then for each \( 0 < \varepsilon < 1 \) there exists \( n_1, n_2 \in N \) such that
\[ n \geq n_1 \implies \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon, \tag{9} \]
and
\[ m \geq n_2 \Rightarrow \mathcal{M}(x_m, x_m, x, t) > 1 - \varepsilon. \] (10)
If set \( n_0 = \max\{n_1, n_2\} \), then for every \( n, m \geq n_0 \) we have:
\[
\mathcal{M}(x_n, x_n, x_m, t) \geq \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(x_n, x_n, x, t) \\
> (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon).
\]
By taking the limit when \( \varepsilon \to 0 \) in above inequality we get \( \mathcal{M}(x_n, x_n, x_m, t) \geq 1 \).
Hence \( \{x_n\} \) is a Cauchy sequence.

**Lemma 3.9** Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space. If there exist sequences \(\{x_n\}\) and \(\{y_n\}\) such that \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\), then
\[
\lim_{n \to \infty} \mathcal{M}(x_n, x_n, y_n, t) = \mathcal{M}(x, y, t). \] (11)

**Proof 8** Since \(\lim_{n \to \infty} x_n = x\) and \(\lim_{n \to \infty} y_n = y\), then for each \(0 < \varepsilon < 1\) there exist \(n_1, n_2 \in \mathbb{N}\) such that
\[ \forall \ n \geq n_1 \Rightarrow \mathcal{M}(x_n, x_n, x, t) > 1 - \varepsilon, \] (12)
and
\[ \forall \ n \geq n_2 \Rightarrow \mathcal{M}(y_n, y_n, y, t) > 1 - \varepsilon. \] (13)
If set \( n_0 = \max\{n_1, n_2\} \), then for every \( n \geq n_0 \) we have:
\[
\mathcal{M}(x_n, x_n, y_n, t) \\
\geq \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(y_n, y_n, x, t) \\
\geq \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(x_n, x_n, x, t) \ast \mathcal{M}(y_n, y_n, y, t) \ast \mathcal{M}(y_n, y_n, y, t) \ast \mathcal{M}(x, x, t) \\
> (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast \mathcal{M}(x, x, t).
\]
By taking the limit when \( \varepsilon \to 0 \) in above inequality we get
\[
\mathcal{M}(x_n, x_n, y_n, t) \geq \mathcal{M}(x, y, t). \] (14)

On the other hand, we have
\[
\mathcal{M}(x, x, y, t) \\
\geq \mathcal{M}(x, x, x, t) \ast \mathcal{M}(x, x, y, t) \ast \mathcal{M}(y, y, x, t) \\
\geq \mathcal{M}(x, x, x, t) \ast \mathcal{M}(x, x, x, t) \ast \mathcal{M}(y, y, y, t) \ast \mathcal{M}(y, y, y, t) \ast \mathcal{M}(x_n, x_n, y_n, t) \\
> (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast (1 - \varepsilon) \ast \mathcal{M}(x_n, x_n, y_n, t),
\]
as \( \varepsilon \to 0 \) we have
\[
\mathcal{M}(x, x, y, t) > \mathcal{M}(x_n, x_n, y_n, t). \] (15)
Therefore by relations (14) and (15) we have
\[
\lim_{n \to \infty} \mathcal{M}(x_n, x_n, y_n, t) = \mathcal{M}(x, x, y, t). \] (16)
**Lemma 3.10** Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space with \(a \ast b \geq ab\) for all \(a, b \in [0, 1]\). If define \(S : X^3 \rightarrow [0, \infty)\) by \(S(x, y, z) = \int_0^1 \log_\alpha (\mathcal{M}(x, y, z, t)) dt\), then \(S\) is an \(S\)-metric on \(X\) for \(0 < \alpha < 1\).

**Proof 9** It is clear from the definition that \(S(x, y, z)\) is well defined for each \(x, y, z \in X\). (i) \(S(x, y, z) \geq 0\) for all \(x, y, z \in X\) is trivial.

(ii) \(S(x, y, z) = 0 \iff \log_\alpha (\mathcal{M}(x, y, z, t)) = 0\) for all \(t > 0\)
\[ \iff \mathcal{M}(x, y, z, t) = 1\] for all \(t > 0 \iff x = y = z\).

(iv) Since \(\mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, x, a, t) \ast \mathcal{M}(y, y, a, t) \ast \mathcal{M}(z, z, a, t)\)
\[ \geq \mathcal{M}(x, x, a, t) \cdot \mathcal{M}(y, y, a, t) \cdot \mathcal{M}(z, z, a, t)\]

it follows that,
\[ S(x, y, z) = \int_0^1 \log_\alpha (\mathcal{M}(x, y, z, t)) dt \]
\[ \leq \int_0^1 \log_\alpha (\mathcal{M}(x, x, a, t) \cdot \mathcal{M}(y, y, a, t) \cdot \mathcal{M}(z, z, a, t)) dt \]
\[ \leq \int_0^1 \log_\alpha (\mathcal{M}(x, x, a, t)) dt + \int_0^1 \log_\alpha (\mathcal{M}(y, y, a, t)) dt + \int_0^1 \log_\alpha (\mathcal{M}(z, z, a, t)) dt \]
\[ = S(x, x, a) + S(y, y, a) + S(z, z, a)\]

This proves that \(S\) is an \(S\)-metric on \(X\).

The following lemma plays an important role to give fixed point results on a \(\mathcal{M}\)-fuzzy metric space.

**Lemma 3.11** Let \((X, \mathcal{M}, \ast)\) be a \(\mathcal{M}\)-fuzzy metric space.

(a) \(\{x_n\}\) is a Cauchy sequence in \((X, \mathcal{M}, \ast)\) if and only if it is a Cauchy sequence in the \(S\)-metric space \((X, S)\).

(b) A \(\mathcal{M}\)-fuzzy metric space \((X, \mathcal{M}, \ast)\) is complete if and only if the \(S\)-metric space \((X, S)\) is complete.

**Proof 10** First we show that every Cauchy sequence in \((X, \mathcal{M}, \ast)\) is a Cauchy sequence in \((X, S)\). To this end let \(\{x_n\}\) be a Cauchy sequence in \((X, \mathcal{M}, \ast)\). Then \(\lim_{n, m \to \infty} \mathcal{M}(x_n, x_n, x_m, t) = 1\). Since
\[ S(x_n, x_n, x_m) = \int_0^1 \log_\alpha (\mathcal{M}(x_n, x_n, x_m, t)) dt, \]
is an $S$-metric. Hence, we have
\[
\lim_{n,m \to \infty} S(x_n, x_n, x_m) = \lim_{n,m \to \infty} \int_0^1 \log_a(M(x_n, x_n, x_m, t)) dt = 0.
\]

We conclude that $\{x_n\}$ is a Cauchy sequence in $(X, S)$. Next we prove that completeness of $(X, S)$ implies completeness of $(X, M, \ast)$. Indeed, if $\{x_n\}$ is a Cauchy sequence in $(X, M, \ast)$ then it is also a Cauchy sequence in $(X, S)$. Since the $S$-metric space $(X, S)$ is complete we deduce that there exists $y \in X$ such that $\lim_{n \to \infty} S(x_n, x_n, y) = 0$. Therefore,
\[
\int_0^1 \log_a(M(x_n, x_n, y, t)) dt = \lim_{n \to \infty} S(x_n, x_n, y) = 0.
\]

Hence we follow that $\{x_n\}$ is a convergent sequence in $(X, M, \ast)$.

Now we prove that every Cauchy sequence $\{x_n\}$ in $(X, S)$ is a Cauchy sequence in $(X, M, \ast)$. Since $\{x_n\}$ is a Cauchy sequence in $(X, S)$, then
\[
\lim_{n,m \to \infty} S(x_n, x_n, x_m) = \lim_{n,m \to \infty} \int_0^1 \log_a(M(x_n, x_n, x_m, t)) dt = 0.
\]

Hence, $\lim_{n,m \to \infty} M(x_n, x_n, x_m, t) = 1$.

That is, $\{x_n\}$ is a Cauchy sequence in $(X, M, \ast)$.

We shall have established the lemma if we prove that $(X, S)$ is complete if so is $(X, M, \ast)$. Let $\{x_n\}$ be a Cauchy sequence in $(X, S)$. Then $\{x_n\}$ is a Cauchy sequence in $(X, M, \ast)$, and so it is convergent to a point $y \in X$ with
\[
\lim_{n,m \to \infty} M(x_n, x_m, y, t) = 1.
\]

As a consequence we have
\[
\lim_{n,m \to \infty} S(x_n, x_m, y) = \lim_{n,m \to \infty} \int_0^1 \log_a(M(x_n, x_m, y, t)) dt = 0.
\]

Therefore $(X, S)$ is complete.

**Lemma 3.12** Let $(X, M, \ast)$ be a $M$-fuzzy metric space with $a \ast b = \min\{a, b\}$ for all $a, b \in [0, 1]$. We define $S : X^3 \to [0, \infty)$ by $S(x, y, z) = \int_0^1 \cot(\frac{\pi}{2} M(x, y, z, t)) dt$, then $S$ is an $S$-metric on $X$.

**Proof 11** (i) $S(x, y, z) \geq 0$ is trivial.

(ii) $S(x, y, z) = 0 \iff \cot(\frac{\pi}{2} M(x, y, z, t)) = 0$ for all $t > 0$

\[\iff M(x, y, z, t) = 1 \text{ for all } t > 0 \iff x = y = z.\]
(iii) Since \( M(x, y, z, t) \geq M(x, x, a, t) \cdot M(y, y, a, t) \cdot M(z, z, a, t) = \min\{M(x, x, a, t), M(y, y, a, t), M(z, z, a, t)\} \),

and also since \( 0 < \frac{\pi}{2} M(x, y, z, t) \leq \frac{\pi}{2} \) it follows that,

\[
S(x, y, z) = \int_0^1 \cot (\frac{\pi}{2} M(x, y, z, t)) dt \\
\leq \int_0^1 \cot (\frac{\pi}{2} \{M(x, x, a, t) \cdot M(y, y, a, t) \cdot M(z, z, a, t)\}) dt \\
= \int_0^1 \cot (\frac{\pi}{2} \min\{M(x, x, a, t), M(y, y, a, t), M(z, z, a, t)\}) dt \\
= \min\{\int_0^1 \cot (\frac{\pi}{2} M(x, x, a, t)) dt, \int_0^1 \cot (\frac{\pi}{2} M(y, y, a, t)) dt, \int_0^1 \cot (\frac{\pi}{2} M(z, z, a, t)) dt\} \\
\leq \int_0^1 \cot (\frac{\pi}{2} M(x, x, a, t)) dt + \int_0^1 \cot (\frac{\pi}{2} M(y, y, a, t)) dt + \int_0^1 \cot (\frac{\pi}{2} M(z, z, a, t)) dt \\
= S(x, x, a) + S(y, y, a) + S(z, z, a),
\]

that is \( S \) is an \( S \)-metric on \( X \).

**Remark 3.13** Let \( a, b \in (0, 1] \), then it is a standard result that

\[
\text{Arccot}(\min\{a, b\}) \leq \text{Arccot}(a) + \text{Arccot}(b) - \frac{\pi}{4}
\]

**Lemma 3.14** Let \( (X, M, *) \) be a \( M \)-fuzzy metric space with \( a * b = \min\{a, b\} \)
for all \( a, b \in [0, 1] \). We define \( S : X^3 \rightarrow [0, \infty) \) by

\[
S(x, y, z) = \int_0^1 (\frac{4}{\pi} \text{Arccot}(M(x, y, z, t)) - 1) dt,
\]

then \( S \) is an \( S \)-metric on \( X \).

**Proof 12** (i) \( 0 \leq S(x, y, z) < 1 \) is trivial.

(ii) \( S(x, y, z) = 0 \iff \frac{4}{\pi} \text{Arccot}(M(x, y, z, t)) - 1 = 0 \) for all \( t > 0 \)

\[
\iff \text{Arccot}(M(x, y, z, t)) = \frac{\pi}{4} \text{ for all } t > 0.
\]

\[
\iff M(x, y, z, t) = 1 \text{ for all } t > 0 \iff x = y = z.
\]

(iii) Since

\[
M(x, y, z, t) \geq M(x, x, a, t) \cdot M(y, y, a, t) \cdot M(z, z, a, t) = \min\{M(x, x, a, t), M(y, y, a, t), M(z, z, a, t)\},
\]

it follows that,
\[
\arccot(\mathcal{M}(x, y, z, t)) \\
\leq \arccot[\mathcal{M}(x, y, z, t) * \mathcal{M}(y, y, a, t) * \mathcal{M}(z, z, a, t)] \\
= \arccot(\min\{\mathcal{M}(x, x, a, t), \mathcal{M}(y, y, a, t), \mathcal{M}(z, z, a, t)\}) \\
\leq \arccot(\mathcal{M}(x, x, a, t)) + \arccot(\mathcal{M}(y, y, a, t)) + \arccot(\mathcal{M}(z, z, a, t)) - \frac{\pi}{2}
\]
Hence,
\[
S(x, y, z) = \int_0^1 \left(\frac{4}{\pi} \arccot(\mathcal{M}(x, y, z, t)) - 1\right) dt \\
\leq \int_0^1 \left(\frac{4}{\pi} \arccot(\mathcal{M}(x, x, a, t)) - 1\right) dt + \int_0^1 \left(\frac{4}{\pi} \arccot(\mathcal{M}(y, y, a, t)) - 1\right) dt \\
+ \int_0^1 \left(\frac{4}{\pi} \arccot(\mathcal{M}(z, z, a, t)) - 1\right) dt \\
= S(x, x, a) + S(y, y, a) + S(z, z, a),
\]
that is $S$ is an $S$-metric on $X$.

We now apply the theorem 3.1 to prove the following fixed point result in $\mathcal{M}$-fuzzy metric spaces.

**Theorem 3.15** Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $d * f \geq df$ for all $d, f \in [0, 1]$. Let $A, B, C$ and $T$ be self maps on $X$ satisfying:

(i) $A(X) \subseteq T(X), B(X) \subseteq C(X)$ and $T(X)$ or $C(X)$ is a closed subset of $X$;

(ii) there exists positive real numbers $a, b, c, e$ such that $a + b + c + 3e < 1$ and for each $x, y, z \in X$,

\[
\mathcal{M}(Ax, Ay, Bz, t) \geq \mathcal{M}^a(Cx, Cx, Tz, t) * \mathcal{M}^b(Cx, Cx, Ax, t) \\
* \mathcal{M}^e(Tz, Tz, Bz, t) * [\mathcal{M}(Cx, Cx, Bz, t) * \mathcal{M}(Tz, Tz, Ay, t)]^e.
\]

(iii) the pairs $(A, C)$ and $(B, T)$ are weakly compatible.

Then $A, B, C$ and $T$ have a unique common fixed point in $X$.

**Proof 13** From inequality (ii) above, we get,
\[
\int_0^1 \log_{\alpha}^{\mathcal{M}(Ax, Ay, Bz, t)} dt \\
\leq \int_0^1 \left(\frac{\mathcal{M}^a(Cx, Cx, Tz, t) * \mathcal{M}^b(Cx, Cx, Ax, t) \\
* \mathcal{M}^e(Tz, Tz, Bz, t) * [\mathcal{M}(Cx, Cx, Bz, t) * \mathcal{M}(Tz, Tz, Ay, t)]^e}{\alpha}\right) dt.
\]
\[
\begin{align*}
&\leq \int_0^1 \log_\alpha \left( M^a(Cx,Cx,Tz,t)M^b(Cx,Cx,Ax,t) \\
&\quad \quad \quad + \int_0^1 \log_\alpha M(Tz,Tz,Bz,t)[M^c(Cx,Cx,Bz,t)M^e(Tz,Tz,Ay,t)] \right) dt \\
&= a \int_0^1 \log_\alpha M(Cx,Cx,Tz,t) dt + b \int_0^1 \log_\alpha M(Cx,Cx,Ax,t) dt \\
&\quad + c \int_0^1 \log_\alpha M(Tz,Tz,Bz,t) dt + e \left( \int_0^1 \log_\alpha M(Tz,Tz,Ay,t) dt \right). \\
\end{align*}
\]

If set \( S(x,y,z) = \int_0^1 \log_\alpha M(x,y,z,t) dt \) for every \( x, y, z \in X \) and \( 0 < \alpha < 1 \). Then it follows that,

\[
S(Ax, Ay, Bz) \leq aS(Cx, Cx, Tz) + bS(Cx, Cx, Ax) + cS(Tz, Tz, Bz) \\
+ e(S(Cx, Cx, Bz) + S(Tz, Tz, Ay)).
\]

Hence by Lemma 3.14 all of conditions Theorem 3.1 hold. Thus \( A, B, C \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 3.16** Let \( (X, M, *) \) be a complete \( M \)-fuzzy metric space with \( d*f \geq df \) for all \( d, f \in [0,1] \). Let \( A \) and \( B \) be self maps on \( X \) satisfying: there exists positive real numbers \( a, b, c, e \) such that \( a + b + c + 3e < 1 \) and for each \( x, y, z \in X \),

\[
M(Ax, Ay, Bz, t) \geq M^a(x, x, z, t) * M^b(x, x, Ax, t) \\
\quad \quad \quad * M^c(z, z, Bz, t) * [M(x, x, Bz, t) * M(z, z, Ay, t)]e.
\]

Then \( A \) and \( B \) have a unique common fixed point in \( X \).

**Proof 14** It is enough set \( T = C = I \), identity map in Theorem 3.15.

**References**


Received: November 23, 2017