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Regularity of Weak Solutions to Some Anisotropic Elliptic Equations

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Abstract

We consider boundary value problem of the form

$$\begin{cases} \sum\limits_{i=1}^{n} D_i(a_i(x, Du(x))) = f, & x \in \Omega, \\ u(x) = u_*(x), & x \in \partial\Omega. \end{cases}$$

We show that regularity of boundary datum u_* forces u to have regularity as well. A similar result is obtained for obstacle problem.

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1 Introduction

Let Ω be a bounded open subset of R^n, $n \geq 2$. We consider the elliptic equation

$$\sum_{i=1}^{n} D_i(a_i(x, Du(x))) = f, \tag{1.1}$$

where a_i: $\Omega \times R^n \to$ R with $x \to a_i$(x,z) continuous and satisfying

$$|a_i(x, z)| \leq c(1 + \sum_{j=1}^{n} |z_j|^{p_j})^{1-\frac{1}{p_i}}, \quad i = 1, 2, \cdots, n, \tag{1.2}$$

and

$$\nu \sum_{i=1}^{n} |z_i - \tilde{z}_i|^{p_i} \leq \sum_{i=1}^{n} (a_i(x,z) - a_i(x,\tilde{z}))(z_i - \tilde{z}_i), \tag{1.3}$$

for some positive constant ν. For $p_1,\cdots, p_n \in (1,+\infty)$, let $\overline{p} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$ and $p_i' = \frac{p_i}{p_i-1}$ be the harmonic mean of $p_1,\cdots,p_n$ and the Hölder conjugate of p_i, respectively. In this paper we assume $\overline{p} < n$ and we introduce the Sobolev exponent $\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$. The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined as usual by

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1,\ldots,n \right\}$$

and $W_0^{1,(p_i)}(\Omega)$ is denoted to be the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,(p_i)}(\Omega)$. We refer to [1,2] for the theory of these spaces. The word *anisotropic* means that the exponent p_i of the derivative $D_i v = \frac{\partial v}{\partial x_i}$ might be different from the exponent p_j of the derivative $D_j v$ when $i \neq j$. For some recent developments on anisotropic functionals and anisotropic elliptic equations and systems, see[3-5].

We work in Marcinkiewicz spaces: if $q > 1$, then the space $M^m(\Omega)$ consists of measurable functions g on Ω such that

$$\sup_{t>0} t|\{x \in \Omega : |g(x)| > t\}|^{\frac{1}{m}} < \infty.$$

This condition is equivalently stated as

$$|||g(x)|||_m = \sup_{E\subset\Omega, |E|>0} \frac{1}{|E|^{\frac{1}{m'}}} \int_E |g(x)|dx < \infty.$$

We recall that $L^m(\Omega)$ is a proper subspace of $M^m(\Omega)$, and if $g \in M^m(\Omega)$ for some $m > 1$, then $g \in L^{m-\varepsilon}(\Omega)$ for every $0 < \varepsilon \leq q-1$.

It is well known that there exists a positive constant c, depending only on Ω, such that

$$\|v\|_{L^r(\Omega)} \leq c \prod_{i=1}^{n} \|D_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{n}}, \qquad \forall r \in [1, \overline{p}^*], \tag{1.4}$$

for any v$\in W_0^{1,(p_i)}(\Omega)$. In the following the letter c will freely denote a constant, not necessarily the same in any two occurrences, while only the relevant dependence will be highlighted.

Let $T_k(u)$ is the usual truncation of u at level k >0, that is,

$$T_k(u) = \max\{-k, \min\{k, u\}\}.$$

Moreover, let

$$G_k(u) = u - T_k(u).$$

In [6] Agnese Di Castro considered the following problem

$$\begin{cases} -\sum\limits_{i=1}^{n} D_i[|D_i u|^{p_i-2} D_i u] = f, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

and gave some results concerning existence and regularity of weak solutions of (1.5).

In this paper we present some results concerning the case of f belonging to a Marcinkiewicz space, M^m of (1.1), in the case of $m > (\bar{p}^*)'$, where $p_\infty = \max\{p_n, \bar{p}^*\}$, $p_n = \max\{p_i\}$.

2 Main Results

These are the main results of the paper.

Theorem 2.1 *Let $f \in M^m(\Omega)$, $m > (\overline{p}^*)'$, $u_* \in W^{1,1}(\Omega)$ with $D_i u_* \in M^{p_i m}$, $i = 1, 2, \cdots, n$, and under previous assumptions (1.2)-(1.3), let u be a weak solution for the problem (1.1), that is*

$$\int_\Omega \sum_{i=1}^{n} a_i(x, Du(x)) D_i v(x) dx = \int_\Omega f v dx, \quad \forall v \in W_0^{1,(p_i)}(\Omega). \tag{2.6}$$

i) If $m > \frac{n}{\overline{p}}$, then u-u_* is bounded;
ii) If $m = \frac{n}{\overline{p}}$, then there exists a constant $\beta > 0$ such that

$$\int_\Omega e^{\beta|u-u_*|} < \infty;$$

iii) If $(\overline{p}^*)' < m < \frac{n}{\overline{p}}$, then $u - u_*$ belongs to M^s with

$$s = \frac{m\overline{p}^*(\overline{p}-1)}{m\overline{p} + \overline{p}^* - m\overline{p}^*} = \frac{mn(\overline{p}-1)}{n - m\overline{p}}. \tag{2.7}$$

We also consider obstacle problem for the elliptic equation (1.1). Let

$$\mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega) = \left\{ v \in W^{1,(p_i)}(\Omega) : v \geq \psi, \text{ a.e. } \Omega, \text{ and } v - u_* \in W_0^{1,(p_i)}(\Omega) \right\},$$

where for the boundary datum u_* and the obstacle function ψ, we assume that

$$u_*, \psi \in W^{1,1}(\Omega), D_i u_*, D_i \psi \in M^{p_i m}(\Omega), \text{ for every } i = 1, \cdots, n, \tag{2.8}$$

The next theorem shows that higher integrability of $\theta = \max\{\psi, u_*\}$ forces solutions $u \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$ to be more integrable.

Theorem 2.2 *Let $f \in M^m(\Omega)$, and under the assumptions (1.2)-(1.3) and (2.8), let $u \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$ be a solution to obstacle problem for the elliptic equation (1.1), that is*

$$\int_\Omega \sum_{i=1}^N a_i(x,Du(x))\cdot(D_iv(x)-D_iu(x))dx \geq \int_\Omega \sum_{i=1}^N f^i(x)\cdot(v(x)-u(x))dx, \quad \forall v \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega). \tag{2.9}$$

i) If m> $\frac{n}{\overline{p}}$, then u-θ is bounded;
ii) If m= $\frac{n}{\overline{p}}$, then there exists a constant $\beta > 0$ such that

$$\int_\Omega e^{\beta|u-\theta|} < \infty;$$

iii) If $(\overline{p}^*)' < m < \frac{n}{\overline{p}}$, then $u-\theta$ belongs to M^s, with s satisfies (2.7).

3 Proof of the Theorems.

Proof of Theorem 2.1. We take

$$v = G_k(u-u_*) = \begin{cases} u-u_*-k, & u-u_*>k, \\ 0, & |u-u_*| \leq k, \\ u-u_*+k, & u-u_*<-k \end{cases}$$

in (2.6) and we have

$$\sum_{i=1}^n \int_\Omega a_i(x,Du(x))D_iG_k(u-u_*) = \int_\Omega fG_k(u-u_*).$$

This implies

$$\sum_{i=1}^n \int_{A_k} a_i(x,Du(x))D_i(u-u_*) = \int_{A_k} f(u-u_*-k\ \mathrm{sign}(u-u_*)),$$

where $A_k = \{|u - u_*| > k\}$. Hence by (1.2), (1.3) and Young inequality we obtain that

$$\begin{aligned} & \nu \sum_{i=1}^{n} \int_{A_k} |D_i u - D_i u_*|^{p_i} \\ \leq\ & \sum_{i=1}^{n} \int_{A_k} (a_i(x, Du) - a_i(x, Du_*))(D_i u - D_i u_*) \\ =\ & \int_{A_k} f(u - u_* - k \text{ sign}(u - u_*)) - \sum_{i=1}^{n} \int_{A_k} a_i(x, Du_*)(D_i u - D_i u_*) \\ \leq\ & \int_{A_k} |f||u - u_*| + \sum_{i=1}^{n} \int_{A_k} |a_i(x, Du_*)||D_i u - D_i u_*| \\ \leq\ & \int_{A_k} |f||u - u_*| + c \sum_{i=1}^{n} \int_{A_k} (1 + \sum_{j=1}^{N} |D_i u_*|^{p_j})^{1-\frac{1}{p_i}} (D_i u - D_i u_*) \\ \leq\ & \int_{A_k} |f||u - u_*| + c(\varepsilon) \sum_{i=1}^{n} \int_{A_k} (1 + \sum_{i=1}^{n} |D_i u_*|^{p_i}) + \varepsilon \sum_{i=1}^{n} \int_{A_k} |D_i u - D_i u_*|^{p_i}, \end{aligned} \tag{3.10}$$

where we have used the fact

$$|f(u - u_* - k \text{ sign}(u - u_*))| \leq |f||u - u_*|.$$

The last term in the right hand side of (3.10) is absorbed by the left hand side, provided ε is small enough. Then

$$\int_{A_k} |D_i u - D_i u_*|^{p_i} \leq c \left(\int_{A_k} |f||u - u_*| + |A_k| + \sum_{i=1}^{n} \int_{A_k} |D_i u_*|^{p_i} \right). \tag{3.11}$$

Therefore, by (1.4), with $r = \overline{p}^*$, Hölder inequality and (3.11), we get

$$\begin{aligned} & \left(\int_{A_k} |u - u_*|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}} \\ \leq\ & c \prod_{i=1}^{n} \left(\int_{A_k} |D_i u - D_i u_*|^{p_i} \right)^{\frac{1}{p_i n}} \\ \leq\ & c \left(\int_{A_k} |f||u - u_*| + |A_k| + \sum_{i=1}^{n} \int_{A_k} |D_i u_*|^{p_i} \right)^{\frac{1}{\overline{p}}} \\ \leq\ & c \left[\left(\int_{A_k} |f|^{(\overline{p}^*)'} \right)^{\frac{1}{(\overline{p}^*)'}} \left(\int_{A_k} |u - u_*|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}} + |A_k| + \sum_{i=1}^{n} \int_{A_k} |D_i u_*|^{p_i} \right]^{\frac{1}{\overline{p}}}. \end{aligned} \tag{3.12}$$

Since $f \in M^m(\Omega)$ and $D_i u_* \in M^{p_i m}(\Omega)$, and $m \geq (\overline{p}^*)'$, we have

$$\int_{A_k} |f|^{(\overline{p}^*)'} \leq c|A_k|^{1-\frac{(\overline{p}^*)'}{m}}, \quad \int_{A_k} |D_i u_*|^{p_i} \leq c|A_k|^{1-\frac{1}{m}}.$$

Then by applying Young inequality and $|A_k|^{\frac{1}{m}} \leq |\Omega|^{\frac{1}{m}}$, (3.12) becomes

$$\begin{aligned}
& c(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{\overline{p}}{\overline{p}^*}} \\
\leq\ & (|A_k|^{1-\frac{(\overline{p}^*)'}{m}})^{\frac{1}{(\overline{p}^*)'}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + |A_k| + |A_k|^{1-\frac{1}{m}} \\
\leq\ & |A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + \frac{1}{k}|A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}} \cdot |A_k|^{\frac{1}{m}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} \\
& + \frac{1}{k}|A_k|^{-\frac{1}{m}} \int_{A_k} |u-u_*| \\
\leq\ & |A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} + c|A_k|^{\frac{1}{(\overline{p}^*)'}-\frac{1}{m}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}} \\
& + c|A_k|^{-\frac{1}{m}}(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{1}{\overline{p}^*}}|A_k|^{\frac{1}{(\overline{p}^*)'}} \\
\leq\ & c(\varepsilon)|A_k|^{(\frac{1}{(\overline{p}^*)'}-\frac{1}{m})(\overline{p})'} + \varepsilon(\int_{A_k} |u-u_*|^{\overline{p}^*})^{\frac{\overline{p}}{\overline{p}^*}}.
\end{aligned}$$

Hence by applying Hölder inequality with exponents $\overline{p}^*$ and $(\overline{p}^*)'$ to $\int_\Omega |G_k(u-u_*)| = \int_{A_k} |u-u_*|$ and by simplifying, we obtain

$$\int_\Omega |G_k(u-u_*)| \leq c|A_k|^{(\frac{1}{(\overline{p}^*)'}-\frac{1}{m})\frac{1}{\overline{p}-1}+1-\frac{1}{\overline{p}^*}}. \tag{3.13}$$

We define $g(k) = \int_\Omega |G_k(u-u_*)|$ and we recall that $g'(k)$=$-|A_k|$, for almost every k (see[7], [8]). We obtain, from (3.13), that

$$g(k)^{\frac{1}{\gamma}} \leq -cg'(k),$$

with $\gamma = (\frac{1}{(\overline{p}^*)'} - \frac{1}{m})\frac{1}{\overline{p}-1} + 1 - \frac{1}{\overline{p}^*}$. Therefore

$$1 \leq -cg'(k)g(k)^{-\frac{1}{\gamma}} = -\frac{c}{1-\frac{1}{\gamma}}(g(k)^{1-\frac{1}{\gamma}})'. \tag{3.14}$$

If we are in case i) of Theorem 1, we note that

$$1 - \frac{1}{\gamma} > 0.$$

Therefore, by integrating (3.14) from 0 to k, we get

$$k \leq -c[g(k)^{1-\frac{1}{\gamma}} - g(0)^{1-\frac{1}{\gamma}}],$$

i.e.

$$cg(k)^{1-\frac{1}{\gamma}} \leq -k + c\|u-u_*\|_{L^1(\Omega)}^{1-\frac{1}{\gamma}}.$$

Since $g(k)$ is a non-negative and decreasing function, from the latter inequality we deduce that there exists k_0, such that g(k_0)=0, and so $u - u_* \in L^\infty(\Omega)$. In case ii) of Theorem 2.1, since $m = \frac{n}{\overline{p}}$, $\gamma = 1$, we have

$$1 \leq -c\frac{g'(x)}{g(x)}.$$

By integrating from 0 to k, we have

$$\frac{k}{c} \leq \log[\frac{\|u-u_*\|_{L^1(\Omega)}}{g(k)}],$$

and since the function $t \to e^t$ increases, we obtain

$$e^{\frac{k}{c}} \leq \frac{\|u-u_*\|_{L^1(\Omega)}}{g(k)} \Rightarrow g(k)e^{\frac{k}{c}} \leq \|u-u_*\|_{L^1(\Omega)}.$$

So, recalling that

$$g(k) = \int_\Omega |G_k(u-u_*)| \geq \int_{A_{2k}} |G_k(u-u_*)| \geq k|A_{2k}|, \tag{3.15}$$

Hence, if $k \geq 1$, we have

$$g(k) \geq |A_{2k}| \Rightarrow |A_{2k}|e^{\frac{k}{c}} \leq \|u-u_*\|_{L^1(\Omega)}.$$

Hence, if $k \geq 2$, we get

$$|A_k|e^{\frac{k}{2c}} \leq \|u-u_*\|_{L^1(\Omega)}. \tag{3.16}$$

We prove now that the previous inequality implies that

$$\sum_{k=0}^{+\infty} e^{k\beta}|A_k| < \infty,$$

with $0 < \beta < \frac{1}{2c}$. Indeed, by (3.16),

$$\sum_{k=0}^{+\infty} e^{k\beta}|A_k| \leq (1+e)|\Omega| + \sum_{k=2}^{+\infty} \frac{\|u-u_*\|_{L^1(\Omega)}}{e^{k(\frac{1}{2c}-\beta)}} < \infty.$$

Since

$$\sum_{k=0}^{+\infty} e^{k\beta}|A_k| < +\infty \Rightarrow \int_\Omega e^{\beta|u-u_*|} < +\infty,$$

ii) is proved. To conclude, we consider case iii). In this case we have

$$1 - \frac{1}{\gamma} < 0.$$

Therefore,

$$1 \leq c(g(k)^{1-\frac{1}{\gamma}})'.$$

By integration from 0 to k, we obtain

$$k \leq c[g(k)^{1-\frac{1}{\gamma}} - g(0)^{1-\frac{1}{\gamma}}] \leq cg(k)^{1-\frac{1}{\gamma}},$$

and so

$$g(k)^{-1+\frac{1}{\gamma}} \le \frac{c}{k} \Rightarrow g(k) \le \frac{c}{k^{\frac{\gamma}{1-\gamma}}}.$$

Therefore, by (3.16), it holds true that

$$|A_{2k}| \le \frac{g(k)}{k} \le \frac{c}{k^{\frac{\gamma}{1-\gamma}}k} = \frac{c}{k^{\frac{1}{1-\gamma}}}.$$

By recalling the definition of γ, we obtain

$$\frac{1}{\gamma} = \frac{mn(\overline{p}-1)}{n-m\overline{p}} = s,$$

so that $u - u_* \in M^s(\Omega)$. This ends the proof of Theorem 2.1.

Proof of Theorem 2.2. Let $u \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$ be a solution to obstacle problem for the (1.1). For $k \in (0,+\infty)$ we define

$$v' = \theta + T_k(u-\theta).$$

We now show that $v' \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$. For the first case $u-\theta > k$, one has $v'=\theta+k \ge \theta \ge \psi$, for the second case $|u-\theta| \le k$, we obviously have $\psi \le u = v'$; for the third case $u-\theta < -k$, we have $\psi \le u < v' = \theta-k$. Since $u \in u_*+W_0^{1,(p_i)}$ and $u \ge \psi$, a.e. Ω, then $\theta = \max\{\psi, u_*\} = u_* = u$ on $\partial\Omega$, thus $v'=0$ on $\partial\Omega$. This implies $v' = u$ on $\partial\Omega$, and therefore $v' \in \mathcal{K}^{(p_i)}_{\psi,u_*}(\Omega)$ and v' satisfied (2.9).

Take $v'(x)$ as the test function, the next proof is similar to the proof of Theorem 2.1 with θ in place of u_*.

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