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Ordered (L, e)-filters

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Abstract

We introduce the notion of ordered (L, e)-filters with fuzzy partially order e on complete residuated lattice L. We define the images and preimages of (L, e)-filters using Zadeh image and preimage operators. We study the images and preimages of (L, e)-filters induced by functions. We investigate their properties.

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1 Introduction

Höhle *et al.* [5,6] introduced the notion of L-filter on a complete quasi-monoidal lattice (including GL-monoid [4]) L instead of a completely distributive lattice ([2-4]) as an extension of fuzzy filters [1,2]. The notion of L-filter facilitated to study L-fuzzy topologies [3,5,6], L-fuzzy uniform spaces [5,6] and topological structures [7].

In this paper, we define ordered (L, e)-filters with fuzzy partially order e on complete residuated lattice L and investigate their properties. We consider the Zadeh image operator $\phi_L^{\rightarrow}$ and the Zadeh preimage operator $\phi_L^{\leftarrow}$ in a sense [8]. We investigate the images and preimages of (L, e)-filters induced by functions.

2 Preliminaries

Definition 2.1 [5,6,9] A triple $(X, \leq, *)$ is called a *complete residuated lattice* iff it satisfies the following properties:

(L1) $(X, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(X, *, 1)$ is a commutative monoid;
(L3) $*$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) * b = \bigvee_{i\in\Gamma}(a_i * b).$$

Example 2.2 [5,6,9] (1) Each frame $(L, \leq, \wedge)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0,1], \leq, t)$, is a complete residuated lattice.

(3) Define a binary operation $*$ on $[0,1]$ by $x * y = \max\{0, x+y-1\}$. Then $([0,1], \leq, *)$ is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^c : L \to L$ defined by $a^c = a \to 0$ is called a *strong negation* if $a^{cc} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^c)$ is a complete residuated lattice with a strong negation ^{c}.

Definition 2.3 [5,6,9] Let X be a set. A function $e_X : X \times X \to L$ is called *fuzzy partially order* on X if it satisfies the following conditions:
(E1) $e_X(x,x) = 1$ for all $x \in X$,
(E2) $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$, for all $x, y, z \in X$,
(E3) if $e_X(x,y) = e_X(y,x) = 1$, then $x = y$.
The pair (X, e_X) is a *fuzzy partially order set* (simply, fuzzy poset).

Let $(X, \leq, *)$ be a complete residuated lattice. A fuzzy poset (X, e_X) is a *p-fuzzy poset* if $e_X(x_1, y_1) \odot e_X(x_2, y_2) \leq e_X(x_1 * x_2, y_1 * y_2)$ for each $x_i, y_i \in X$ and $e_X(x,y) = 1$ if $x \leq y$.

Lemma 2.4 [5,6,9] *For each $x, y, z, x_i, y_i \in L$, we define $x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then the following properties hold.*
(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$ and $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*
(2) $x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y$.
(3) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$
(4) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(5) $x \to (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \to y_i)$
(6) $(\bigwedge_{i\in\Gamma} x_i) \to y = \bigvee_{i\in\Gamma}(x_i \to y)$.
(7) $\bigwedge_{i\in\Gamma} y_i^c = (\bigvee_{i\in\Gamma} y_i)^c$ *and* $\bigvee_{i\in\Gamma} y_i^c = (\bigwedge_{i\in\Gamma} y_i)^c$.
(8) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(9) $1 \to x = x$.
(10) *If $x \leq y$, then $x \to y = 1$.*
(11) $(x \to y) \odot (y \to z) \leq x \to z$.
(12) $(x_1 \to y_1) \odot (x_2 \to y_2) \leq (x_1 \odot x_2 \to y_1 \odot y_2)$.

Example 2.5 (1) We define a map $e_L : L \times L \to L$ $e_L(x,y) = x \to y = \bigvee\{z \in L \mid x \odot z \leq y\}$. Then (L, e_L) is a p-fuzzy poset from Lemma 2.4 (10-12).

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(f,g) = \bigwedge_{x \in X}(f(x) \to g(x))$. Then (L^X, e_{L^X}) is a p-fuzzy poset.

(3) If (X, e_X) is a fuzzy poset and we define a function $e_X^{-1}(x,y) = e_X(y,x)$, then (X, e_X^{-1}) is a fuzzy poset.

3 Ordered (L, e)-filters

Definition 3.1 Let $(X, \leq, *)$ be a complete residuated lattice and e_X a fuzzy poset. A mapping $\mathcal{F} : X \to L$ is called an *ordered* (L, e_X)*-filter* (for short, (L, e_X)-filter) on X if it satisfies the following conditions:

(F1) $\mathcal{F}(0) = 0$ and $\mathcal{F}(1) = 1$,

(F2) $\mathcal{F}(x * y) \geq \mathcal{F}(x) \odot \mathcal{F}(y)$, for each $x, y \in X$,

(F3) $\mathcal{F}(x) \odot e_X(x,y) \leq \mathcal{F}(y)$.

The pair $(X, \mathcal{F})$ is called an (L, e_X)*-filter space*.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (L, e)-filters on X. We say $\mathcal{F}_1$ is *finer* than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is *coarser* than $\mathcal{F}_1$) iff $\mathcal{F}_2 \leq \mathcal{F}_1$.

Example 3.2 (1) We define a fuzzy poset $e_L(x,y) = x \to y$ as in Example 2.5(1). Let $\mathcal{F}$ be an (L, e_L)-filter on L. By (F3), since $\mathcal{F}(x) \odot e_X(x,0) \leq \mathcal{F}(0) = 0$, we have $\mathcal{F}(x) \leq x^{cc} = x$. Also, since $x = \mathcal{F}(1) \odot e_L(1,x) \leq \mathcal{F}(x)$, we have $\mathcal{F}(x) = x$.

(2) Since $x \leq y$ iff $e_X(x,y) = 1$, by (F3), If $x \leq y$, then $\mathcal{F}(x) \leq \mathcal{F}(y)$. Hence the above definition is an extension of Höhle *et al.* [14,15].

(3) Let $(X, \leq, *)$ be a complete residuated lattice and (X, e_X) a p-fuzzy poset with $e_X(1,0) = 0$. Then a mapping $\mathcal{F} : X \to L$ defined by $\mathcal{F}(x) = e_X(1,x)$ is an (L, e_X)-filter on X because

(F1) $\mathcal{F}(0) = e_X(1,0) = 0, \mathcal{F}(1) = e_X(1,1) = 1$,

(F2) $\mathcal{F}(x * y) = e_X(1, x * y) \geq e_X(1,x) \odot e_X(1,y) = \mathcal{F}(x) \odot \mathcal{F}(y)$.

(F3) $\mathcal{F}(x) \odot e_X(x,y) = e_X(1,x) \odot e_X(x,y) \leq e_X(1,y) = \mathcal{F}(y)$.

Theorem 3.3 *Let (X, e_X) be a p-fuzzy poset. A mapping $\mathcal{F} : X \to L$ is an (L, e_X)-filter on X iff it satisfies the conditions (F1),(F3) and*

$$\mathcal{F}(x \Rightarrow y) \odot \mathcal{F}(x) \leq \mathcal{F}(y)$$

where $x \Rightarrow y = \bigvee\{z \in X \mid x * z \leq y\}$.

Proof

$$\begin{aligned} \mathcal{F}(y) &\geq \mathcal{F}(x * (x \Rightarrow y)) \odot e_X(x * (x \Rightarrow y), y) \\ &\geq \mathcal{F}(x * (x \Rightarrow y)) \geq \mathcal{F}(x) \odot \mathcal{F}(x \Rightarrow y). \end{aligned}$$

$$\begin{aligned}\mathcal{F}(x*y) &\geq \mathcal{F}(x \Rightarrow (x*y)) \odot \mathcal{F}(x)\\ &\geq \mathcal{F}(y) \odot e_X(y, x \Rightarrow (x*y)) \odot \mathcal{F}(x) = \mathcal{F}(x) \odot \mathcal{F}(y).\end{aligned}$$

Theorem 3.4 *Let $(X, \leq, *)$ be a complete residuated lattice and (X, e_X) a p-fuzzy poset. If $\mathcal{H} : X \to L$ is a function satisfying the following condition:*
(C) *$\mathcal{H}(1) = 1$ and for every finite index set K,*

$$\bigvee_K \odot_{i\in K}\mathcal{H}(x_i) \odot e_X(*_{i\in K}x_i, 0) = 0.$$

We define a function $\mathcal{F}_{\mathcal{H}} : L^X \to L$ as

$$\mathcal{F}_{\mathcal{H}}(x) = \bigvee(\odot_{i\in K}\mathcal{H}(x_i)) \odot e_X(*_{i\in K}x_i, x)$$

where the $\bigvee$ is taken for every finite set K.
Then:
(1) *$\mathcal{F}_{\mathcal{H}}$ is an (L, e_X)-filter on X,*
(2) *if $\mathcal{H} \leq \mathcal{F}$ and $\mathcal{F}$ is an (L, e_X)-filter on X, then $\mathcal{F}_{\mathcal{H}} \leq \mathcal{F}$.*

Proof. (1) (F1) By the condition (C), $\mathcal{F}_{\mathcal{H}}(1) = 1$ and $\mathcal{F}_{\mathcal{H}}(0) = 0$.
(F2) For each two finite index sets K and J,

$$\begin{aligned}&\mathcal{F}_{\mathcal{H}}(x_1) \odot \mathcal{F}_{\mathcal{H}}(x_2)\\ &= \bigvee_K \Big((\odot_{i\in K}\mathcal{H}(y_i)) \odot e_X(*_{i\in K}y_i, x_1)\Big)\\ &\odot \bigvee_J \Big((\odot_{j\in J}\mathcal{H}(z_j)) \odot e_X(*_{j\in J}z_j, x_2)\Big)\\ &\leq \bigvee_{K\cup J} \Big((\odot_{i\in K}\mathcal{H}(y_i)) \odot (\odot_{j\in J}\mathcal{H}(z_j)) \odot e_X((*_{i\in K}y_i) * (*_{j\in J}z_j), x_1 * x_2)\Big)\\ &\leq \mathcal{F}_{\mathcal{H}}(x_1 * x_2).\end{aligned}$$

(F3)

$$\mathcal{F}_{\mathcal{H}}(x) \odot e_X(x, y) = \bigvee(\odot_{i\in K}\mathcal{H}(y_i)) \odot e_X(*_{i\in K}y_i, x) \odot e_X(x, y) \leq \mathcal{F}_{\mathcal{H}}(y).$$

Thus, $\mathcal{F}_{\mathcal{H}}$ is an (L, e_X)-filter on X.
(2) For each finite index set K, we have

$$\begin{aligned}\mathcal{F}(x) &\geq \mathcal{F}(*_{i\in K}x_i) \odot e_X(*_{i\in K}x_i, x)\\ &\geq \odot_{i\in K}\mathcal{F}(x_i) \odot e_X(*_{i\in K}x_i, x)\\ &\geq \odot_{i\in K}\mathcal{H}(x_i) \odot e_X(*_{i\in K}x_i, x).\end{aligned}$$

Thus $\mathcal{F}_{\mathcal{H}} \leq \mathcal{F}$.

Definition 3.5 Let $(X,\mathcal{F})$ and $(Y,\mathcal{G})$ be two (L,e_X) and (L,e_Y)-filter spaces. Then a function $\phi : X \to Y$ is said to be:

(1) an *filter map* iff $\mathcal{G}(y) \leq \bigvee_{x\in\phi^{-1}(\{y\})} \mathcal{F}(x)$, for all $y \in Y$,

(2) an *filter preserving map* iff $\mathcal{F}(x) \leq \mathcal{G}(\phi(x))$ for all $x \in X$.

(3) an *ordered preserving map* iff $e_X(x,y) \leq e_Y(\phi(x),\phi(x))$ for all $x,y \in X$.

(4) $\phi^{-1} : Y \to X$ is an *ordered preserving relation* iff for all $x,y \in Y$,

$$e_Y(x,y) \leq \bigwedge_{a\in\phi^{-1}(\{x\}),b\in\phi^{-1}(\{y\})} e_X(a,b).$$

Naturally, the composition of filter maps (resp. filter preserving maps) is an filter map (resp. filter preserving map).

Example 3.6 Let $X = \{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$ be a set, $(L = [0,1],\odot)$ complete residuated lattice with $x \odot y = (x+y-1)\vee 0$ and $x \to y = (1-x+y)\wedge 1$. Define functions $\mathcal{F}_i : X \to [0,1]$ as follows:

$$\mathcal{F}_1(x) = \begin{cases} 1 \text{ if } & x = 1,\\ \frac{1}{2} \text{ if } & x = \frac{3}{4},\\ 0 \text{ otherwise,} \end{cases} \quad \mathcal{F}_2(x) = \begin{cases} 1 \text{ if } & x = 1,\\ \frac{3}{4} \text{ if } & x \in \{\frac{3}{4},\frac{1}{2}\}\\ 0 \text{ otherwise,} \end{cases}$$

$e_0, e_1 : X \times X \to [0,1]$ as follows:

$$e_0(x,y) = \begin{cases} 1 \text{ if } & x \leq y,\\ 0 \text{ otherwise,} \end{cases}$$

and $e_1(x,y) = x \to y$.

(1) $\mathcal{F}_1$ is an (L,e_0)-filter but not an (L,e_1)-filter because

$$\frac{1}{2} = (\mathcal{F}_1(1) \odot e_1(1,\frac{1}{2})) \not\leq \mathcal{F}_1(\frac{1}{2}) = 0.$$

Since $\mathcal{F}_1(x) \odot e_1(x,0) = 0$, we obtain $\mathcal{F}_{\mathcal{F}_1}(x) = e_1(1,x) \vee (\frac{1}{2} \odot e_1(\frac{3}{4},x)) = x$.

(2) Since $0 = \mathcal{F}_2(\frac{1}{2}\odot\frac{1}{2}) \not\geq \mathcal{F}_2(\frac{1}{2})\odot\mathcal{F}_2(\frac{1}{2}) = \frac{1}{2}$, $\mathcal{F}_2$ is neither an (L,e_0)-filter nor an (L,e_1)-filter. Furthermore, it satisfies the condition(C) of Theorem 3.4 because

$$\mathcal{F}_2(\frac{1}{2}) \odot \mathcal{F}_2(\frac{1}{2}) \odot e_i(\frac{1}{2}\odot\frac{1}{2},0) = \frac{1}{2} \neq 0,\ i \in \{0,1\}.$$

(3) Let $X = \{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$ be a set. Define a function $\phi : X \to Y$ as follows:

$$\phi(0) = \phi(\frac{1}{4}) = 0,\ \phi(\frac{1}{2}) = \phi(\frac{3}{4}) = \frac{1}{2}, \phi(1) = 1.$$

Let $\mathcal{G} : Y \to [0,1]$ be an $([0,1],e_1)$ filter as

$$\mathcal{G}(y) = \begin{cases} 1 \text{ if } & y = 1,\\ \frac{1}{2} \text{ if } & y = \frac{1}{2},\\ 0 \text{ if } & y = 0. \end{cases}$$

Since $\mathcal{F}_1$ and $\mathcal{G}$ are $([0,1],e_0)$ and $([0,1],e_1)$ filters, respectively and $\mathcal{G}(y) \leq \bigvee_{x\in\phi^{-1}(\{y\})} \mathcal{F}_1(x)$ and $\mathcal{F}_1(x) \leq \mathcal{G}(\phi(x))$, ϕ are filter map and filter preserving map. Since $e_0(x,y) \leq e_1(\phi(x),\phi(x))$ for all $x,y \in X$, $\phi : (X,e_0) \to (Y,e_1)$ is an order preserving map. Since $\frac{3}{4} = e_1(1,\frac{3}{4}) \not\leq e_1(\phi(1),\phi(\frac{3}{4})) = \frac{1}{2}$, then $\phi : (X,e_1) \to (Y,e_1)$ is not an order preserving map. Since $\frac{1}{2} = e_1(\frac{1}{2},0) \not\leq e_1(\phi(\frac{3}{4}),0) = \frac{1}{4}$, for $\frac{1}{4} \in \phi^{-1}(\frac{1}{2}), 0 \in \phi^{-1}(0)$, then $\phi^{-1} : (Y,e_1) \to (X,e_1)$ is not an order preserving relation.

4 The preimages and images of (L,e)-filters

In this section we consider the preimages and images of (L,e_X)-filters.

Definition 4.1 Let $\phi : X \to Y$ be a function, $\mathcal{F}$ an (L,e_X)-filter on X and $\mathcal{G}$ an (L,e_Y)-filter Y.

(1) The *image* of $\mathcal{F}$ is a function $\phi_L^{\rightarrow}(\mathcal{F}) : Y \to L$ defined by

$$\phi_L^{\rightarrow}(\mathcal{F})(y) = \bigvee\{\mathcal{F}(x) \mid x = \phi^{-1}(y)\}.$$

(2) The *preimage* of $\mathcal{G}$ is a function $\phi_L^{\leftarrow}(\mathcal{G}) : X \to L$ defined by

$$\phi_L^{\leftarrow}(\mathcal{G})(x) = \mathcal{G}(\phi(x)).$$

(3) Let $\mathcal{H} : X \to L$ be a function and $x \in X$. We denote

$$[\mathcal{H}](x) = \bigvee_{y\in X} \mathcal{H}(y) \odot e_X(y,x).$$

Theorem 4.2 *Let $(X,\leq,*)$ and $(Y,\leq,\star)$ be complete residuated lattices. Let $\phi : X \to Y$ be an order preserving function with $\phi(x*y) \geq \phi(x)\star\phi(y)$, $\phi(0) = 0$ and $\phi(1) = 1$, e_X, e_Y p-fuzzy posets and $\mathcal{G}$ an (L,e_Y)-filter on Y. Then:*

(1) *$[\phi_L^{\leftarrow}(\mathcal{G})]$ is the coarsest (L,e_X)-filter for which $\phi : (X,[\phi_L^{\leftarrow}(\mathcal{G})]) \to (Y,\mathcal{G})$ is a filter map.*

(2) *If $e_X(x,y) = e_Y(\phi(x),\phi(y))$ for $x,y \in X$, then $[\phi_L^{\leftarrow}(\mathcal{G})] = \phi_L^{\leftarrow}(\mathcal{G})$.*

Proof. (1) (F1) is obvious.

$$\begin{aligned}[\phi_L^{\leftarrow}(\mathcal{G})](0) &= \textstyle\bigvee_{x\in X} \phi_L^{\leftarrow}(\mathcal{G})(x) \odot e_X(x,0)\\ &\leq \textstyle\bigvee_{x\in X} \mathcal{G}(\phi(x)) \odot e_Y(\phi(x),0) \leq \mathcal{G}(0) = 0.\end{aligned}$$

(F2)

$$\begin{aligned}
&[\phi_L^{\leftarrow}(\mathcal{G})](x_1) \odot [\phi_L^{\leftarrow}(\mathcal{G})](x_2) \\
&= \bigvee_{z_1\in X}(\phi_L^{\leftarrow}(\mathcal{G})(z_1) \odot e_X(z_1,x_1)) \odot \bigvee_{z_2\in X}(\phi_L^{\leftarrow}(\mathcal{G})(z_2) \odot e_X(z_2,x_2)) \\
&= \bigvee_{z_1\in X}(\mathcal{G}(\phi(z_1)) \odot e_X(z_1,x_1)) \odot \bigvee_{z_2\in X}(\mathcal{G}(\phi(z_2)) \odot e_X(z_2,x_2)) \\
&\leq \bigvee_{y_1,y_2\in X}(\mathcal{G}(\phi(z_1)\star\phi(z_2)) \odot e_X(z_1*z_2, x_1*x_2)) \\
&\quad (\mathcal{G}(\phi(z_1)\star\phi(z_2)) \odot e_Y(\phi(z_1)\star\phi(z_2), \phi(z_1*z_2)) \leq \mathcal{G}(\phi(z_1*z_2))) \\
&\leq \bigvee_{y_1,y_2\in X}(\mathcal{G}(\phi(z_1*z_2)) \odot e_Y(\phi(z_1*z_2), \phi(x_1*x_2)) \leq \mathcal{G}(\phi(x_1*x_2)) \\
&= [\phi_L^{\leftarrow}(\mathcal{G})](x_1*x_2).
\end{aligned}$$

(F3)

$$\begin{aligned}
&[\phi_L^{\leftarrow}(\mathcal{G})](x) \odot e_X(x,z) \\
&= \bigvee_{w\in X}(\phi_L^{\leftarrow}(\mathcal{G})(w) \odot e_X(w,x) \odot e_X(x,z)) \\
&\leq \bigvee_{w\in X}(\phi_L^{\leftarrow}(\mathcal{G})(w) \odot e_X(w,z)) = [\phi_L^{\leftarrow}(\mathcal{G})](z).
\end{aligned}$$

$$\begin{aligned}
\bigvee_{x\in\phi^{-1}(\{y\})}[\phi_L^{\leftarrow}(\mathcal{G})](x) &= \bigvee_{x\in\phi^{-1}(\{y\})}\bigvee_{z\in X}\phi_L^{\leftarrow}(\mathcal{G})(z) \odot e_X(z,x) \\
&\geq \bigvee_{x\in\phi^{-1}(\{y\})}\phi_L^{\leftarrow}(\mathcal{G})(x) \odot e_X(x,x) = \mathcal{G}(y).
\end{aligned}$$

Let $\phi : (X,\mathcal{F}) \to (Y,\mathcal{G})$ be a filter map;i.e. $\mathcal{G}(y) \leq \bigvee_{x\in\phi^{-1}(\{y\})}\mathcal{F}(x)$.

$$\begin{aligned}
[\phi_L^{\leftarrow}(\mathcal{G})](x) &= \bigvee_{z\in X}\phi_L^{\leftarrow}(\mathcal{G})(z) \odot e_X(z,x) \\
&= \bigvee_{z\in X}\mathcal{G}(\phi(z)) \odot e_X(z,x) \\
&\leq \bigvee_{w\in\phi^{-1}(\{\phi(z)\})}\mathcal{F}(w) \odot e_X(w,x) \leq \mathcal{F}(x).
\end{aligned}$$

(2) It follows from

$$\begin{aligned}
[\phi_L^{\leftarrow}(\mathcal{G})](x) &= \bigvee_{z\in X}\phi_L^{\leftarrow}(\mathcal{G})(z) \odot e_X(z,x) \\
&= \bigvee_{z\in X}\mathcal{G}(\phi(z)) \odot e_Y(\phi(z),\phi(x)) \\
&= \mathcal{G}(\phi(x)) = \phi_L^{\leftarrow}(\mathcal{G})(x).
\end{aligned}$$

Example 4.3 Let $X = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $Y = \{0, \frac{1}{2}, 1\}$ be sets, $(L = [0,1], \odot)$ complete residuated lattice as in Example 3.6. Define functions $\mathcal{G}_1 : Y \to [0,1], \mathcal{F}_1, \mathcal{F}_2 : X \to [0,1]$ as follows:

$$\mathcal{G}_1(y) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{1}{2} \text{ if } & y = \frac{1}{2}, \\ 0 \text{ otherwise,} \end{cases} \quad \mathcal{F}_1(x) = x, \quad \mathcal{F}_2(x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x = \frac{3}{4} \\ 0 \text{ otherwise.} \end{cases}$$

e_0, e_1 are fuzzy posets on X and Y as follows:

$$e_0(x,y) = \begin{cases} 1 \text{ if } & x \leq y, \\ 0 \text{ otherwise,} \end{cases}$$

and $e_1(x,y) = x \to y$.

(1) Define a function $\phi : X \to Y$ as follows:

$$\phi(0) = \phi(\frac{1}{4}) = 0,\ \phi(\frac{1}{2}) = \phi(\frac{3}{4}) = \frac{1}{2}, \phi(1) = 1.$$

We have $\phi(x \odot y) \geq \phi(x) \odot \phi(y)$, but not equality as follows:

$$\frac{1}{2} = \phi(\frac{3}{4} \odot \frac{3}{4}) > \phi(\frac{3}{4}) \odot \phi(\frac{3}{4}) = 0.$$

Furthermore, $e_0(x,y) \leq e_0(\phi(x), \phi(y))$. Then $[\phi_L^{\leftarrow}(\mathcal{G}_1)] = \phi_L^{\leftarrow}(\mathcal{G}_1)$ is (L, e_0)-filters as

$$[\phi_L^{\leftarrow}(\mathcal{G}_1)](x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x \in \{\frac{1}{2}, \frac{3}{4}\} \\ 0 \text{ if } & x \in \{\frac{1}{4}, 0\}. \end{cases}$$

But $\phi_L^{\leftarrow}(\mathcal{G}_1)$ is not a (L, e_1)-filter because

$$\frac{3}{4} = \phi_L^{\leftarrow}(\mathcal{G}_1)(1) \odot e_1(1, \frac{3}{4}) \not\leq \phi_L^{\leftarrow}(\mathcal{G}_1)(\frac{3}{4}) = \frac{1}{2}$$

But $[\phi_L^{\leftarrow}(\mathcal{G}_1)](x) = e_1(1,x) \vee (\frac{1}{2} \odot e_1(\frac{3}{4}, x)) \vee (\frac{1}{2} \odot e_1(\frac{1}{2}, x))$ is an (L, e_1)-filter as follows

$$[\phi_L^{\leftarrow}(\mathcal{G}_1)](x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{3}{4} \text{ if } & x = \frac{3}{4}, \\ \frac{1}{2} \text{ if } & x = \frac{1}{2}, \\ \frac{1}{4} \text{ if } & x = \frac{1}{4}, \\ 0 \text{ if } & x = 0. \end{cases}$$

Since $\frac{3}{4} = e_1(1, \frac{3}{4}) \not\leq e_1(\phi(1), \phi(\frac{3}{4})) = e_1(1, \frac{1}{2}) = \frac{1}{2}$, The converse of the above theorem need not be true.

(2) Define a function $\psi : Y \to X$ as follows:

$$\psi(0) = 0,\ \psi(\frac{1}{2}) = \frac{1}{2}, \psi(1) = 1.$$

Since $e_i(x,y) = e_i(\psi(x), \psi(y))$ for $i \in \{0,1\}$, We obtain $[\psi_L^{\leftarrow}(\mathcal{F}_i)] = \psi_L^{\leftarrow}(\mathcal{F}_i)$ for $i \in \{1,2\}$ as follows;

$$\psi_L^{\leftarrow}(\mathcal{F}_1)(x) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{1}{2} \text{ if } & y = \frac{1}{2} \\ 0 \text{ if } & y = 0. \end{cases} \quad \psi_L^{\leftarrow}(\mathcal{F}_2)(y) = \begin{cases} 1 \text{ if } & y = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.4 *Let $(X, \leq, *)$ and $(Y, \leq, \star)$ be complete residuated lattices. Let $\phi : X \to Y$ be a function with $\phi(x * y) \leq \phi(x) \star \phi(y)$ with $\phi(1) = 1$ and $\phi(0) = 0$, e_X, e_Y p-fuzzy posets. Let $\mathcal{F}$ and $\mathcal{G}$ be (L, e_X) and (L, e_Y)-filters, respectively. Then we have the following properties.*

(1) *If $\mathcal{F}(x)\odot e_Y(\phi(x),0)=0$, then $[\phi_L^{\rightarrow}(\mathcal{F})]$ is the coarsest (L,e_Y)-filter for which $\phi:(X,\mathcal{F})\to(Y,[\phi_L^{\rightarrow}(\mathcal{F})])$ is a filter preserving map.*

(2) *If ϕ is injective and ϕ^{-1} is order-preserving relation, $[\phi_L^{\rightarrow}(\mathcal{F})]$ is an (L,e_X)-filter.*

(3) *If ϕ is surjective, ϕ^{-1} is order-preserving relation and $\mathcal{F}$ is an (L,e_X)-filter with $\mathcal{F}(x)\odot e_Y(\phi(x),0)=0$, then $\phi_L^{\rightarrow}(\mathcal{F})$ is an (L,e_X)-filter.*

(4) *If $\phi:X\to Y$ is an order preserving map with $\phi(x*y)=\phi(x)\star\phi(y)$, then $[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G})])]$ is an (L,e_Y)-filter on Y with $[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G})])]\le\mathcal{G}$.*

Proof. (1) (F1) Since $\phi^{-1}(1)=1$, $[\phi_L^{\rightarrow}(\mathcal{F})](1)=1$. By the assumption,

$$\begin{aligned}[\phi_L^{\rightarrow}(\mathcal{F})](0) &= \bigvee_{y\in Y}\phi_L^{\rightarrow}(\mathcal{F})(y)\odot e_Y(y,0)\\ &= \bigvee_{x=\phi^{-1}(\{y\})}\mathcal{F}(x)\odot e_Y(\phi(x),0)\le 0.\end{aligned}$$

(F2) Since

$$\begin{aligned}&e_Y(\phi(x_1*x_2),\phi(x_1)\star\phi(x_2))\odot e_Y(\phi(x_1)\star\phi(x_2),z_1\star z_2)\\ &\le e_Y(\phi(x_1*x_2),z_1\star z_2),\end{aligned}$$

by the definition of $\phi_L^{\rightarrow}(\mathcal{F})(y_i)$ for $i\in\{1,2\}$ and (L4), there exist $x_i\in L^Y$ with $x_i=\phi^{-1}(y_i)$ such that

$$\begin{aligned}&[\phi_L^{\rightarrow}(\mathcal{F})](z_1)\odot[\phi_L^{\rightarrow}(\mathcal{F})](z_2)\\ &=\Big(\bigvee_{y_1\in X}\phi_L^{\rightarrow}(\mathcal{F})(y_1)\odot e_Y(y_1,z_1)\Big)\odot\Big(\bigvee_{y_2\in X}\phi_L^{\rightarrow}(\mathcal{F})(y_2)\odot e_Y(y_2,z_2)\Big)\\ &=\Big(\bigvee_{x_1\in X}\mathcal{F}(x_1)\odot e_Y(\phi(x_1),z_1)\Big)\odot\Big(\bigvee_{x_2\in X}\mathcal{F}(x_2)\odot e_Y(\phi(x_2),z_2)\Big)\\ &\le\bigvee_{x_1,x_2\in X}\Big(\mathcal{F}(x_1*x_2)\odot e_Y(\phi(x_1)\star\phi(x_2),z_1\star z_2)\Big)\\ &\le\bigvee_{x_1,x_2\in X}\Big(\mathcal{F}(x_1*x_2)\odot e_Y(\phi(x_1*x_2),z_1\star z_2)\Big)\\ &\le[\phi_L^{\rightarrow}(\mathcal{F})](z_1\star z_2)\end{aligned}$$

(F3)

$$\begin{aligned}[\phi_L^{\rightarrow}(\mathcal{F})](y)\odot e_Y(y,z) &=\Big(\bigvee_{w\in Y}\phi_L^{\rightarrow}(\mathcal{F})(w)\odot e_Y(w,y)\Big)\odot e_Y(y,z)\\ &\le\bigvee_{w\in Y}\phi_L^{\rightarrow}(\mathcal{F})(w)\odot e_Y(w,z)=[\phi_L^{\rightarrow}(\mathcal{F})](z).\end{aligned}$$

Hence $[\phi_L^{\rightarrow}(\mathcal{F})]$ is an (L,e_Y)-filter on X. Moreover, $\phi:(X,\mathcal{F})\to(Y,[\phi_L^{\rightarrow}(\mathcal{F})])$ is a filter preserving map from:

$$\begin{aligned}[\phi_L^{\rightarrow}(\mathcal{F})](\phi(x)) &=\bigvee_{w\in Y}\phi_L^{\rightarrow}(\mathcal{F})(w)\odot e_Y(w,\phi(x))\\ &\ge\phi_L^{\rightarrow}(\mathcal{F})(\phi(x))\odot e_Y(\phi(x),\phi(x))\ge\mathcal{F}(x).\end{aligned}$$

Let $\phi:(X,\mathcal{F})\to(Y,\mathcal{G})$ be a filter preserving map. For each $y\in Y$, we have

$$\begin{aligned}[\phi_L^{\rightarrow}(\mathcal{F})](y) &=\bigvee_{z\in Y}\phi_L^{\rightarrow}(\mathcal{F})(z)\odot e_Y(z,y)\\ &\le\bigvee_{z\in Y}\Big(\bigvee_{x\in\phi^{-1}(\{z\})}(\mathcal{F}(x)\odot e_Y(z,y))\Big)\\ &\le\bigvee_{\phi(x)\in Y}\Big(\mathcal{G}(\phi(x))\odot e_Y(\phi(x),y)\Big)\\ &\le\mathcal{G}(y).\end{aligned}$$

(2) Since ϕ is injective, $\phi^{-1}(\phi(\{x\})) = \{x\}$,

$$\begin{array}{ll} \mathcal{F}(x) \odot e_Y(\phi(x), 0) & \leq \mathcal{F}(x) \odot \bigvee_{z \in \phi^{-1}(\phi(\{x\})), w \in \phi^{-1}(\{0\})} e_X(z, w) \\ & = \mathcal{F}(x) \odot e_X(x, 0) \leq \mathcal{F}(0) = 0. \end{array}$$

(3) Since ϕ is surjective, $\phi(\phi^{-1}(\{y\})) = \{y\}$. Thus $\phi_L^{\rightarrow}(\mathcal{F}) = [\phi_L^{\rightarrow}(\mathcal{F})]$ is an (L, e_X)-filter from:

$$\begin{array}{ll} [\phi_L^{\rightarrow}(\mathcal{F})](y) & = \bigvee_{z \in Y} \phi_L^{\rightarrow}(\mathcal{F})(z) \odot e_Y(z, y) \\ & \leq \bigvee_{z \in Y} \bigvee_{x \in \phi^{-1}(\{z\}), w \in \phi^{-1}(\{y\})} \mathcal{F}(x) \odot e_X(x, w) \\ & \leq \bigvee_{w \in \phi(\{y\})} \mathcal{F}(w) \\ & = \phi_L^{\rightarrow}(\mathcal{F})(y), \end{array}$$

$$[\phi_L^{\rightarrow}(\mathcal{F})](y) \geq \phi_L^{\rightarrow}(\mathcal{F})(y) \odot e_Y(y, y) = \phi_L^{\rightarrow}(\mathcal{F})(y).$$

(4) From the condition of (1), we have

$$\begin{array}{l} [\phi_L^{\leftarrow}(\mathcal{G})](x) \odot e_X(\phi(x), 0) \\ = \bigvee_{w \in X} (\phi_L^{\leftarrow}(\mathcal{G})(w) \odot e_X(w, x) \odot e_X(\phi(x), 0)) \\ \leq \bigvee_{w \in X} (\mathcal{G}(\phi(w)) \odot e_Y(\phi(w), \phi(x)) \odot e_Y(\phi(x), 0)) \\ \leq \bigvee_{w \in X} (\mathcal{G}(\phi(w)) \odot e_Y(\phi(w), 0)) \\ \leq \mathcal{G}(0) = 0. \end{array}$$

Hence $[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G})])]$ exists.

$$\begin{array}{l} [\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G})])](y) \\ = \bigvee_{z \in Y} (\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G})])(z) \odot e_X(z, y)) \\ = \bigvee_{x \in X} \bigvee_{x = \phi^{-1}(\{z\})} ([\phi_L^{\leftarrow}(\mathcal{G})](x) \odot e_X(z, y)) \\ = \bigvee_{x \in X} \bigvee_{x = \phi^{-1}(\{z\})} (\bigvee_{w \in X} \phi_L^{\leftarrow}(\mathcal{G})(w) \odot e_X(w, x) \odot e_X(z, y)) \\ = \bigvee_{x \in X} \bigvee_{x = \phi^{-1}(\{z\})} (\bigvee_{w \in X} \mathcal{G}(\phi(w)) \odot e_Y(\phi(w), \phi(x)) \odot e_Y(\phi(x), y)) \\ = \bigvee_{w \in X} (\mathcal{G}(\phi(w)) \odot e_Y(\phi(w), y)) \\ \leq \mathcal{G}(y). \end{array}$$

Example 4.5 Let $X = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $Y = \{0, \frac{1}{2}, 1\}$ be sets, $L = [0, 1]$ the complete residuated lattice as in Example 3.6. Define functions $\mathcal{G}_i : Y \to [0, 1]$ as follows:

$$\mathcal{G}_1(y) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x = \frac{1}{2}, \\ 0 \text{ otherwise}, \end{cases} \quad \mathcal{G}_2(x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x = \frac{3}{4} \\ 0 \text{ otherwise}, \end{cases}$$

e_0, e_1 are fuzzy posets on X and Y as follows:

$$e_0(x, y) = \begin{cases} 1 \text{ if } & x \leq y, \\ 0 \text{ otherwise}, \end{cases}$$

and $e_1(x,y) = x \to y$.

(1) Define a function $\phi : X \to Y$ as follows:

$$\phi(0) = 0, \phi(\frac{1}{4}) = \phi(\frac{1}{2}) = \frac{1}{2}, \phi(\frac{3}{4}) = \phi(1) = 1.$$

Then $\phi(x \odot y) \leq \phi(x) \odot \phi(y)$ and ϕ^{-1} is an order-preserving relation. Let $\mathcal{F}(x) = x$ be an $([0,1], e_1)$ filter on X with $\mathcal{F}(x) \odot e_1(\phi(x), 0) = 0$. Then we obtain an $([0,1], e_1)$ filter $[\phi_L^{\rightarrow}(\mathcal{F})] = \phi_L^{\rightarrow}(\mathcal{F})$ on Y as follows:

$$[\phi_L^{\rightarrow}(\mathcal{F})](y) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{1}{2} \text{ if } & y = \frac{1}{2} \\ 0 \text{ if } & y = 0. \end{cases}$$

(2) Define a function $\psi : X \to Y$ as follows:

$$\psi(0) = \psi(\frac{1}{4}) = 0, \psi(\frac{1}{2}) = \psi(\frac{3}{4}) = \frac{1}{2}, \psi(1) = 1.$$

Then $\frac{1}{2} = \psi(\frac{3}{4} \odot \frac{3}{4}) \not\leq \psi(\frac{3}{4}) \odot \psi(\frac{3}{4}) = 0$. Let $\mathcal{F}(x) = x$ be an $([0,1], e_1)$ filter on X with $\mathcal{F}(\frac{3}{4}) \odot e_1(\psi(\frac{3}{4}), 0) = \frac{1}{4} \neq 0$. Then $([0,1], e_1)$ filter $[\phi_L^{\rightarrow}(\mathcal{F})] = \psi_L^{\rightarrow}(\mathcal{F})$ is not $([0,1], e_1)$ filter on Y as follows:

$$[\psi_L^{\rightarrow}(\mathcal{F})](y) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{3}{4} \text{ if } & y = \frac{1}{2} \\ \frac{1}{4} \text{ if } & y = 0. \end{cases}$$

(3) Define an injective function $\psi : Y \to X$ as follows:

$$\psi(0) = 0, \ \psi(\frac{1}{2}) = \frac{1}{2}, \psi(1) = 1.$$

Define an $([0,1], e_1)$ filter $\mathcal{G}(x) = x$. Then we obtain an $([0,1], e_1)$ filter $[\psi_L^{\rightarrow}(\mathcal{G})](y) = y$.

Example 4.6 Let X, Y, $L = [0,1]$, $\mathcal{G}_1, \mathcal{F}_i$,e_0, e_1 ϕ and ψ be as same in Example 4.3.

(1) $[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G}_1)])]$ is an (L, e_0)-filter as

$$[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G}_1)])](y) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{1}{2} \text{ if } & y = \frac{1}{2}, \\ 0 \text{ if } & y = 0. \end{cases}$$

(2) Since $\frac{3}{4} = e_1(1, \frac{3}{4}) \not\leq e_1(\phi(1), \phi(\frac{3}{4})) = e_1(1, \frac{1}{2}) = \frac{1}{2}$, then $[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G}_1)])]$ is not an (L, e_1) filter such that

$$[\phi_L^{\rightarrow}([\phi_L^{\leftarrow}(\mathcal{G}_1)])](y) = \begin{cases} 1 \text{ if } & y = 1, \\ \frac{3}{4} \text{ if } & y = \frac{1}{2}, \\ \frac{1}{2} \text{ if } & y = 0. \end{cases}$$

(3)

$$\psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))(x) = \begin{cases} 1 \text{ if } & x = 1, \\ \frac{1}{2} \text{ if } & x = \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases} \quad \psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_2))(x) = \begin{cases} 1 \text{ if } & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))$ is not a (L, e_1)-filter because

$$\frac{3}{4} = \psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))(1) \odot e_1(1, \frac{3}{4}) \not\leq \psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))(\frac{3}{4}) = 0$$

But $[\psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))](x) = x = e_1(1, x) \vee (\frac{1}{2} \odot e_1(\frac{1}{2}, x))$ is an (L, e_1)-filter with $[\psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_1))] = \mathcal{F}_1$. Moreover, $[\psi_L^{\rightarrow}(\psi_L^{\leftarrow}(\mathcal{F}_2))] < \mathcal{F}_2$.

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