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Optimal Convex Combination Bounds for The Square Root Mean

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Abstract

The optimal value of parameters α and β are obtained to make the following double inequality holds for all $a, b > 0$ with $a \neq b$,

$$\alpha A(a,b) + (1-a)C(a,b) < Q(a,b) < \beta A(a,b) + (1-\beta)C(a,b)$$

where A(a,b), C(a,b) and Q(a,b) denote arithmetic mean,the ontraharmonic mean,the square root mean of two different positive numbers a and b respectively.

Mathematics Subject Classification: xxxxx

Keywords: The arithmetic mean , The contraharmonic mean , The square root mean.

1 Introduction

For $p \in R$,the power mean of order p of two positive numbers a and b is defined by When $p \neq 0$,

$$M_p(a,b) = ((a^p + b^p)/2)^{1/p},$$

when $p = 0$,

$$M_p(a,b) = \sqrt{ab}.$$

Recently, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for $M_p(a,b)$ can be found in literatures [1-12]. It is well known that $M_p(a,b)$ is continuous and increasingly with respect to $p \in R$. For fixed a and b.If we denote $H(a,b) = 2ab/(a+b)$,$G(a,b) =$

$\sqrt{ab}$,$L(a,b)=(b-a)/(logb-loga)$,$P(a,b)=(a-b)/[4arctan\sqrt{a/b}-\pi]$,$I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$,$T(a,b)=(a-b)/[2arcsin(a-b)/(a+b)]$,$Q(a,b)=\sqrt{(a^2+b^2)/2}$,$C(a,b)=(a^2+b^2)/(a+b)$,

Then

$$\begin{aligned} & min\{a,b\}<H(a,b)<G(a,b)<L(a,b)<P(a,b)\\ < \ & I(a,b)<A(a,b)<T(a,b)<Q(a,b)<C(a,b)<max\{a,b\} \end{aligned}$$

In [13], Alzer and Janous established the following sharp double inequality (see also [14]):

$$M_{log2/log3}(a,b)\le \frac{2}{3}A(a,b)+\frac{1}{3}G(a,b)\le M_{2/3}(a,b)$$

for all $a,b>0$.

In [15], Mao proved

$$M_{1/3}(a,b)\le \frac{1}{3}A(a,b)+\frac{2}{3}G(a,b)\le M_{1/2}(a,b)$$

for all $a,b>0$ and $M_{1/3}(a,b)$ is the best possible lower power mean bound for the sum $(1/3)A(a,b)+2/3G(a,b)$.

2 Monotonicity Theorem

Theorem 2.1 *The double inequality*

$$\alpha A(a,b)+(1-\alpha)C(a,b)<Q(a,b)<\beta A(a,b)+(1-\beta)C(a,b)$$

holds for all $a,b>0$ *if and only if* $\alpha\ge 2-\sqrt{2}$ *and* $\beta\le 1/2$ *.*

Proof Firstly, we prove that

$$Q(a,b)<1/2A(a,b)+1/2C(a,b) \tag{1}$$

$$Q(a,b)>(2-\sqrt{2})A(a,b)+(\sqrt{2}-1)C(a,b) \tag{2}$$

for all $a,b>0$ with $a\neq b$. Without loss of generality, we assume $a>b$. Let $t=a/b>1$ and $P\in\{1/2,2-\sqrt{2}\}$. Then

$$\begin{aligned} & Q(a,b)-[pA(a,b)-(1-p)C(a,b)]\\ = \ & Q(t,1)-[pA(t,1)-(1-p)C(t,1)]\\ = \ & \sqrt{(t^2+1)/2}-[p(t+1)^2+2(1-p)(t^2+1)]/2(t+1)\\ = \ & [\frac{\sqrt{2(t^2+1)}(t+1)}{p(t+1)^2+2(1-p)(t^2+1)}-1][p(t+1)^2+2(1-p)(t^2+1)]/2(t+1) \end{aligned} \tag{3}$$

Let

$$f(t) = \frac{\sqrt{2(t^2+1)}(t+1)}{p(t+1)^2 + 2(1-p)(t^2+1)} - 1 \tag{4}$$

Then simple computations lead to

$$\lim_{t\to 1} f(t) = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[\frac{\sqrt{2(t^2+1)}(t+1)}{p(t+1)^2 + 2(1-p)(t^2+1)} - 1] = \frac{\sqrt{2}}{2-p} - 1 \tag{5}$$

$$f'(t) = [\frac{\sqrt{2(t^2+1)}(t+1)}{p(t+1)^2 + 2(1-p)(t^2+1)} - 1]' = \frac{g(t)}{\sqrt{2(t^2+1)}[p(t+1)^2 + 2(1-p)(t^2+1)]^2} \tag{6}$$

where

$$g(t) = (6p-4)t^3 + (4-2p)t^2 + (2p-4)t + 4 - 6p \tag{7}$$

Now we distinguishes with two cases.

Case 1 If $p = 1/2$, then it follows from (7) that

$$g(t) = -t^3 + 3t^2 - 3t + 1 = -(t-1)^3 \tag{8}$$

for all $t > 1$.

Therefore, inequality (1) follows from (3)-(5) and (6) together with (8).

Case 2 If $p = 2 - \sqrt{2}$, then from (7) we have

$$g(1) = 0, \lim_{t\to\infty} g(t) = -\infty \tag{9}$$

$$g'(t) = 3(6p-4)t^2 + 2(4-29)t + 2p - 4 = (18p-12)t^2 + (8-4p)t + 2p - 4 \tag{10}$$

$$g'(1) = 16p - 8 > 0, \lim_{t\to\infty} g'(t) = -\infty \tag{11}$$

$$g''(t) = 2(18p-12)t^2 + (8-4p) = (36p-24)t + 8 - 4p \tag{12}$$

$$g''(1) = 32p - 16 > 0, \lim_{t\to\infty} g''(t) = -\infty \tag{13}$$

$$g'''(t) = 36p - 24 < 0 \tag{14}$$

From (14) we clearly see that $g''(t)$ is strictly decreasing for $t > 1$, which with (13) implies that there exists a constant $\lambda_1 \in (1, +\infty)$ such that $g''(t) > 0$ for and for $t \in (1, \lambda_1)$ and $g''(t) < 0$ for $t \in (1, \lambda_1)$. This implies that $g'(t)$ is strictly increasing for $t \in (1, \lambda_1)$ and strictly decreasing for $t \in (\lambda_1, +\infty)$.

From (12) implies that there exists a constant $\lambda_2 \in (1, +\infty)$ such that $g'(t) > 0$ for $t \in (1, \lambda_2)$ and $g'(t) < 0$ for $t \in (\lambda_2, +\infty)$. This implies that g(t) is strictly increasing for $t \in (1, \lambda_2)$ and strictly decreasing for $t \in (\lambda_2, +\infty)$.

From (9) implies that there exists a constan $\lambda_3 \in (1, +\infty)$ such that $f'(t) > 0$ for $t \in (1, \lambda_3)$ and $f'(t) < 0$ for $t \in (\lambda_3, +\infty)$. This implies that $f(t)$ is strictly increasing for $t \in (1, \lambda_3)$ and strictly decreasing for $t \in (\lambda_3, +\infty)$.

Note that (5) becomes

$$\lim_{t\to\infty} f(t) = \frac{\sqrt{2}}{2-p} - 1 = 0$$

Thus $f(t) > 0$ for all $t > 1$ and (2) follows.

Secondly, we prove that $1/2A(a, b) + 1/2C(a, b)$ is the best possible upper convex combination bound of arithmetic and contraharmonic means for the square root $Q(a, b)$.

If $\beta > 1/2$, the (13) lead to

$$\lim_{t\to 1^+} g''(t) = 32p - 16 > 0 \tag{15}$$

From (15) and the continuity of $g''(t)$ we see that there exists $\delta = \delta(\beta) > 0$ such that

$$g''(t) > 0$$

for $t \in (1, 1 + \delta)$. (4)-(12) imply that

$$f(t) > 0.$$

Therefore,by (3) $Q(t, 1) > \beta A(t, 1) + (1 - \beta)C(t, 1)$ for $t \in (1, 1 + \beta)$.

Finally, we prove that $(2-\sqrt{2})A(a, b)+(\sqrt{2}-1)C(a, b)$ is the best possible lower convex combination bound of arithmetic and contraharmonic means for the square root Q(a,b) .

If $\alpha < 2 - \sqrt{2}$, then from (3) one has

$$\lim_{t\to+\infty} \frac{\alpha A(t, 1) + (1 - \alpha)C(t, 1)}{Q(t, 1)} = \sqrt{2} - \frac{\sqrt{2}}{2}\alpha > 1$$

Inequality (15) implies there exists $X = X(\alpha) > 1$ such that $\alpha A(t, 1) + (1 - \alpha)C(t, 1) > Q(t, 1)$ for $t \in (X, +\infty)$.

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