ON THE STABILITY OF THE FUNCTIONAL EQUATION

\[ f(x + y + z + xy + yz + xz + xyz) = f(x) + f(y) + f(z) + (x + y + xy) f(z) \]
\[ + (y + z + yz) f(x) + (x + z + xz) f(y) \]

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Abstract

In this paper, we study the Hyers - Ulam stability and the Super-stability of the functional equation

\[ f(x + y + z + xy + yz + xz + xyz) = f(x) + f(y) + f(z) + (x + y + xy) f(z) \]
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1 Introduction

In 1940, S.M.Ulam [20] while he was giving a series of lectures in the University of Wisconsin; he raised a question concerning the stability of homomorphism.

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(.,.) \). Given \( \varepsilon > 0 \) does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \). Then a homomorphism \( H : G_1 \rightarrow G_2 \) exists with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \).
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The first partial solution to Ulam’s question was provided by D.H. Hyers [6]. Indeed, he proved the following celebrated theorem.

**Theorem (D.H. Hyers):** Assume that *X* and *Y* are Banach spaces. If a function \( f : X \to Y \) satisfies the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon
\]
for some \( \varepsilon \geq 0 \) and for all \( x \) in *X*, then the limit
\[
a(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)
\]
exist for each \( x \) in *X* and \( a : X \to Y \) is the unique additive function such that
\[
\|f(x) - a(x)\| \leq \varepsilon
\]
for any \( x \in X \), moreover, if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in E \), then \( a \) is linear.

From the above case, we say that the additive functional equation \( f(x+y) = f(x) + f(y) \) has the Hyers-Ulam stability on \( (X, Y) \). D.H. Hyers explicitly constructed the additive function \( a : X \to Y \) directly from the given function \( f \). This method is called a direct method and it is a powerful tool for studying stability of functional equations.

Th.M.Rassias [15] proved the following substantial generalization of the result of Hyers:

**Theorem 1.1** Let \( X \) and \( Y \) be Banach spaces, let \( \theta \in [0, \infty) \), and let \( P \in [0, 1) \). If a functional equation \( f : X \to Y \) satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \left( \|x\|^P + \|y\|^P \right)
\]
for all \( x, y \in X \), then there is a unique additive mapping \( A : X \to Y \)
\[
\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^P} \|f(x)\|^P
\]
for all \( x \in X \). If in addition, \( f(tx) \) is continuous in \( t \) for each fixed \( x \in X \), then \( A \) is linear.

Due to this fact, the cauchy functional equation \( f(x + y) = f(x) + f(y) \) is said to have the Hyers-Ulam-Rassias stability properly on \( (X, Y) \). A number of result concerning stability of different equations can be found in [1, 2, 3, 5, 8]. Consider the following functional equations
\[
f(xy) = xf(y) + yf(x)
\]
(2)
and
\[ f(x^2) = 2xf(x) \] (3)
which define multiplicative derivations and multiplicative Jordan derivations in algebras. It may be observed that real-valued function \( f(x) = x \log x \) be a solution of the functional equation (2) and (3) on the interval \((0, \infty)\). Extend the result of (2), we obtain
\[ f(xyz) = xyf(z) + yzf(x) + xzf(y). \] (4)

During the 34th international Symposium on Functional Equations, Gy. Maksa [13] posed the Hyers-Ulam Stability problem for the functional equation (2) on the interval \((0, 1]\). The first result concerning the superstability of this equation for functions between operator algebras was obtained by P. Semrl [16]. On the other hand, Zs. Pales [14] remarked that the functional equation (2) for real-valued functions on \([1, \infty)\) is stable in the sense of Hyers and Ulam. The Hyers-Ulam Stability of the functional equations
\[ h(rx^2 + 2x) = 2rxh(x) + 2h(x) \] (5)
and
\[ h(x + y + rxy) = h(x) + h(y) + rxh(y) + ryh(x) \] (6)
were invested by E.H.Lee, I.S. Chang and Y.S. Chang [12] relative to a Multiplicative derivation.

A generalized version of the Hyers Ulam Stability and Superstability of the functional Equations
\[ f(x + y - xy) + xf(y) + yf(x) = f(x) + f(y) \] (7)
was investigated by Y.S. Jung [10].

In this paper, we study the Hyers-Ulam Stability and Superstability of the functional equation
\[ f(x + y + z + xy + yz + xz + xyz) = f(x) + f(y) + f(z) + (x + y + xy)f(z) + (y + z + yz)f(x) + (x + z + xz)f(y). \] (8)
Throughout this paper, let \( N \) denote the set of all natural numbers and \( R \) denote the set of all real numbers.

2 Solutions of Equation(8)

In this section, we try to get the general solution of the functional equation (8) in the interval \((-1, \infty)\). Note that the function, \( f(x) = (x + 1) \ln(x + 1) \) is the solution of the functional equation (8) on the interval \((-1, \infty)\).
Theorem 2.1 Let $X$ be a real (or complex) linear space. A function $f : (-1, \infty) \to X$ satisfies the functional equation (8) for all $x \in (-1, \infty)$ if and only if there exists a solution $D : (0, \infty) \to X$ of the functional equation (4) such that $f(x) = D(x + 1)$ for all $x \in (-1, \infty)$.

Proof. Necessity. Define a mapping $D : (0, \infty) \to X$ by $D(x) := f(x - 1)$. We claim that $D$ is a solution of the functional equation (4). Indeed, for all $x, y \in (0, \infty)$, we have

\[
D(xyz) = f(xyz - 1) \\
= f((x - 1) + (y - 1) + (z - 1) + (x - 1)(y - 1) + (y - 1)(z - 1) \\
+ (x - 1)(z - 1) + (x - 1)(y - 1)(z - 1)) \\
= xyD(z) + yzD(x) + xzD(y).
\]

Hence $D$ is a solution of the functional equation (4). From the definition of $D$, we obtain $f(x) = D(x + 1)$ for all $x \in (-1, \infty)$. The sufficiency part is obvious.

3 Hyers - Ulam stability of Equation(8)

In the following Theorem, we state the result due to F.Skof [17] which is concerning the stability of the additive functional equation $f(x + y) = f(x) + f(y)$ on a restricted domain.

Theorem 3.1 Let $X$ be a real (or complex) Banach space. Given $c > 0$, let a mapping $f : [0, c) \to X$ satisfy the inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]

for some $\delta \geq 0$ and for all $x, y \in [0, c)$ with $x + y \in [0, c)$. Then there exists additive mapping $A : R \to X$ such that

\[
\|f(x) - A(x)\| \leq 3\delta
\]

for all $x \in [0, c)$.

We now present our main theorem on the the Hyers - Ulam stability on the interval $(-1, 0]$ of the functional equation (8). The proof is similar to the one given in [19].

Theorem 3.2 Let $X$ be a real (or complex) Banach space, and let $f : (-1, 0] \to X$ be a mapping satisfying the inequality

\[
\|f(x + y + z + xy + yz + xz + xyz) - f(x) - f(y) - f(z) \\
- (x + y + xy)f(z) - (y + z + yz)f(x) - (x + z + xz)f(y)\| \leq \delta
\]

(9)
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for some $\delta > 0$ and for all $x, y \in (-1, 0]$. Then there exists a solution

$H : (-1, 0) \rightarrow X$ of the functional equation (8) such that

$$\|f(x) - H(x)\| \leq (4e)\delta$$

(10)

for all $x, y \in (-1, 0]$.

Proof. Let $g : (-1, 0] \rightarrow X$ be a mapping defined by

$$g(x) = \frac{f(x)}{x + 1}$$

for all $x \in (-1, 0]$. Then, by (8), we observe that $g$ satisfies inequality

$$\|g(x + y + z + xy + yz + xz) - g(x) - g(y) - g(z)\| \leq \frac{\delta}{(x + 1)(y + 1)(z + 1)}$$

for all $x, y \in (-1, 0)$. Let us now define the mapping $F : [0, \infty) \rightarrow X$ by

$$F(-\ln(x + 1)) = g(x)$$

for all $x \in (-1, 0]$, then, by setting $u = -\ln(x + 1)$, $v = -\ln(y + 1)$ and $w = -\ln(z + 1)$, it will lead to

$$\|F(u + v + w) - F(u) - F(v) - F(w)\| \leq \delta e^{u+v+w}$$

(11)

for all $u, v, w \in (0, \infty]$. This means that

$$\|F(u + v + w) - F(u) - F(v) - F(w)\| \leq \delta e^c$$

(12)

for $u, v, w \in [0, c]$ with $u + v + w < c$, where $c > 1$ is an arbitrary given constant.

By using Theorem (9), we see that there exists an additive mapping $A : R \rightarrow X$ such that $\|F(u) - A(u)\| \leq 3\delta e^c$, for all $u \in [0, c)$. If we let $c \rightarrow 1$ in the last inequality, then we obtain

$$\|F(u) - A(u)\| \leq 3e\delta$$

(13)

for all $u \in [0, 1]$. Moreover, from (11) it follows

$$\|F(u + 2) - F(u) - 2F(1)\| \leq \delta e^{u+2}$$

$$\|F(u + 4) - F(u + 2) - 2F(1)\| \leq \delta e^{u+4}$$

$$\vdots$$

$$\|F(u + 2k) - F(u + 2k - 2) - 2F(1)\| \leq \delta e^{u+2k}$$
for all $u \in [0, 1]$ and $k \in N$. Summing up the above inequalities, we obtain

$$
\|F(u + 2k) - F(u) - 2kF(1)\| \leq \delta e^{u+2k}(1 + e^{-2} + e^{-4} + \ldots + e^{-2k+2})
$$

(14)

for all $u \in [0, 1]$ and $k \in N$. From equation (13), we assert that

$$
\|F(v) - A(v)\| \leq \delta e v
$$

(15)

for all $v \in [0, \infty)$. In fact, when $v \geq 0$ and $k \in NU\{0\}$, we arrive that $v - k \in [0, 1]$. Then by (13) and (14), we have

$$
\begin{align*}
\|F(v) - A(v)\| & \leq \|F(v) - F(v - 2k) - 2kF(1)\| \\
& = \|F(v - 2k) - A(v - 2k)\| + \|A(2k) - 2kF(1)\| \\
& \leq \delta e v + 3\delta e + 2k \|A(1) - F(1)\| \\
& \leq \delta e v + 3\delta e + 3\delta e v \\
& \leq \delta e (v + 3(1 + v)) \\
& \leq 4\delta e v.
\end{align*}
$$

Hence, from (15) and using the definition of $F$, it follows that

$$
\|g(x) - A(-\ln(x + 1))\| \leq 4\delta e \frac{e^{-\ln(x+)} \{4\delta e \}}{x + 1}
$$

for all $x \in (-1, 0]$. Again using the definition of $f(x)$, we obtain

$$
\left\| \frac{f(x)}{x + 1} - A(-\ln(x + 1)) \right\| \leq \frac{4\delta e}{x + 1}
$$

(16)

for all $x \in (-1, 0]$. If we put $H(x) = (x + 1)A(-\ln(x + 1))$ for all $x \in (-1, 0]$, using Theorem (2.1) it can be easily verified that $H$ is a solution of the functional equation (8). Using $H(x)$ and equation (16) it will yield that

$$
\|f(x) - H(x)\| \leq (4e)\delta
$$

for all $x \in (-1, 0]$. This proves the equation (10). Hence the proof of the theorem is complete.

### 4 Superstability of Equation (8)

In this section, we will introduce the following Theorem (4.1) due to F. Skof [18] which is essential to prove the main Theorem.
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Theorem 4.1 Let $X$ be a real (or complex) Banach space, and let $c > 0$ be a given constant. Suppose that a mapping $f : \mathbb{R} \to X$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in \mathbb{R}$ with $|x| + |y| > c$. Then there exists a unique additive mapping $A : \mathbb{R} \to X$ such that

$$\|f(x) - A(x)\| \leq 9\delta$$

for all $x \in \mathbb{R}$.

Now let us prove the main theorem of section which is the super stability of the functional equation (8) on the interval $[0, \infty)$.

Theorem 4.2 Let $X$ be a real (or complex) Banach space, and let $f : [0, \infty) \to X$ be a mapping satisfying the inequality

$$\|f(x+y+z+xy+yz+zx+xz+xyz) - f(x) - f(y) - f(z) - (x+y+y)f(z) - (y+z+y)f(x) - (x+z+xz)f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in [0, \infty)$. Then $f$ satisfies the functional equation (8) for all $x, y \in [0, \infty)$.

Proof. Defining the mapping $g : [0, \infty) \to X$ by $g(x) = f(x+1)$ for all $x \in [0, \infty)$ as in the proof of Theorem (3.2) and define the mapping $F : [0, \infty) \to X$ by $F(ln(x+1)) = g(x)$ for all $x \in [0, \infty)$. Taking $u = ln(x+1), v = ln(y+1)$, and $w = ln(z+1)$, we have

$$\|F(u+v+w) - F(u) - F(v) - F(w)\| \leq \delta e^{-(u+v+w)}$$

for all $u, v, w \in [0, \infty)$. From this, we claim that $F$ is additive. From (18) with $\delta_n = \delta e^{-n}(n \in \mathbb{N})$, it gives $\|F(u+v+w) - F(u) - F(v) - F(w)\| \leq \delta_n$ for all $u, v, w \in [0, \infty)$ with $u + v + w > n$.

Now define a mapping $T : \mathbb{R} \to X$ by

$$T(u) = \begin{cases} F(u) & \text{for } u \geq 0 \\ -F(-u) & \text{for } u < 0. \end{cases}$$

From this, we observe that

$$\|T(u+v) - T(u) - T(v)\| \leq \delta_n$$

for all $u, v \in \mathbb{R}$ with $|u| + |v| > n$. Therefore, by Theorem 4.1, there exists a unique additive mapping $A_n : \mathbb{R} \to X$, such that

$$\|T(u) - A_n(u)\| \leq 9\delta_n$$
for all $u \in R$. Let $m, n \in N$ with $n > m$. Then the additive mapping $A_n : R \to X$ satisfies $\|T(u) - A_n(u)\| \leq 9\delta_m$ for all $u \in R$. The uniqueness argument now implies $A_n = A_m$ for all $n \in N$ with $n > m > 0$, and thus $A_1 = A_2 = \ldots = A_n = \ldots$. Taking the limit in (19) as $n \to \infty$, it gives $T = A_n = A_1$ and this shows that $F$ is additive.

Now, according to the definitions of $F$ and $g$, we have $\frac{f(x)}{x + 1} = F(ln(x + 1))$ for all $x \in [0, \infty)$, and hence by using Theorem (2.1) we see that $f$ satisfies the functional equation (8) for all $x, y \in [0, \infty)$. Since $F$ is additive and $D(x) = xF(ln(x))$ $(x \in [1, \infty))$ is a solution of the functional equation (4). This is completes the proof of the theorem.

5 Generalized version of the Hyers-Ulam Stability of Equation(8)

In this section, we are going to investigate a generalized version of the Hyers-Ulam Stability of the followed equation (8) on the interval $[0, 1)$. In order to prove our main Theorem, we need the following definition and proposition which are proved by J. Tabor [19] concerning the stability of the additive functional equation $f(x + y) = f(x) + f(y)$ on the interval $[0, \infty)$.

**Definition.** A function $g : [0, \infty) \to [0, \infty)$ is called exponentially increasing if it is increasing and there exists $\gamma > 1$ and $h \in [0, \infty)$ such that $g(x + h) \geq \gamma g(x)$ for all $x \in [0, \infty)$.

**Proposition 5.1.** Suppose that $g : [0, \infty) \to [0, \infty)$ is exponentially increasing with constants $\gamma$ and $h$ as in Definition, and that $g(0) > 0$.

Let $K = 2^{\frac{g(h)}{g(0)}} + \frac{1}{\gamma - 1}$, and let $f : [0, \infty) \to X$ be an arbitrary function such that

$$f(x + y) - f(x) - f(y) \in g(x + y)V$$

for all $x \in [0, \infty)$. Then there exists a unique additive function $A : [0, \infty) \to X$ such that $A(h) = f(h)$ and that

$$f(x) - A(x) \in Kg(x)V$$

for all $x \in [0, \infty)$.

Throughout this section, we assume that $X$ is a sequentially complete topological vector space and $V$ is a closed convex, bounded and symmetric with respect to zero subset of $X$. The proof of the following Theorem is very analogous to one given in [19].

**Theorem 5.1** Let $f : [0, 1) \to X$ be a function such that

$$f(x + y + z + xy + yz + xz + xyz) - f(x) - f(y) - f(z) - (x + y + xy)f(z) - (y + z + yz)f(x) - (x + z + xz)f(y) \in V$$

(20)
for all $x, y \in [0, 1)$, and let $z \in (0, 1)$ be an arbitrary fixed. Then there exists a unique function $F_z : [0, 1) \to X$ such that

$$F_z(z) = f(z)$$

\[F_z(x + y + z + xy + yz + xz + xyz) - (x + y + xy)F_z(z) - (y + z + yz)F_z(x) - (x + z + xz)F_z(y) = F_z(x) + F_z(y) + F_z(z)\] (22)

and that

$$f(x) - F_z(x) \in K_z V$$

for all $x, y \in [0, 1)$, where $K_z = \frac{2}{1 + z} + \frac{1}{z}$.

**Proof.** Let $K$ be a set of real numbers. By $X^K$ we denote the vector space of all functions from $K$ into $X$. We define the linear operator $B : X^{[0,1)} \to X^{[0,\infty)}$ by the formula $B(f)(x) = \exp_x f(1 + \exp(-x))$ for all $x \in [0, \infty)$. Now from the equation (20), we can show that $f$ also satisfies the following equation

$$B(f)(u + v + w) - B(f)(u) - B(f)(v) - B(f)(w) \in \exp(u + v + w)V$$

for $u, v, w \in [0, \infty)$ and so they are equivalent. Obviously $\exp$ is exponentially increasing with

$h := -\exp^{-1}(1 + z) = -\ln(1 + z) =: \gamma$.

Therefore by Proposition 5.1, there exists a unique

$$A_h(h) = B(f)(h)$$

$$A_h(u + v + w) = A_h(u) + A_h(v) + A_h(w)$$

$$B(f)(u) - A_h(u) \in K_z \exp(u)V$$

for all $x \in [0, \infty)$, where $K_z = \frac{2\exp(h)}{1 + z} + \frac{\gamma}{2 - 1} = \frac{2}{1 + z} + \frac{1}{z}$.

Let $F_z := B^{-1}(A_h)$. Then we can easily verify from (24), (25) and (26) that $F_z$ satisfies (21), (22) and (23), respectively.

Now we claim that $F_z$ is unique. Suppose that there exists $F'_z$ satisfying (24), (25) and (26). Then $B(F'_z)$ satisfies (21), (22) and (23), hence $B(F'_z) = A_h = B(F_z)$. Since $B$ is bijection, this implies that $F'_z = F_z$. Hence the proof of the theorem is complete.

**References**


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