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On the Periodic Solutions of Some Rational Difference Systems II

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Abstract

In this paper, we study the periodic solutions of the rational difference system

$$x_{n+1} = \frac{\beta y_{n-2}}{-\beta - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{\beta x_{n-2}}{-\beta - x_{n-2}y_{n-1}x_n},$$

$$z_{n+1} = \frac{\beta x_{n-2} + \beta y_{n-2}}{-\beta - x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, ...,$$

where $\beta \neq 0$ and the initial conditions are arbitrary real numbers.

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1 Introduction

In 2011, Kurbanli et al. [2] studied the behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary non negative real numbers.

In 2012, Elsayed et al. [1] studied the solutions of the systems of the difference equation

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}z_{n-p}}, \quad y_{n+1} = \frac{x_{n-p}y_{n-p}z_{n-p}}{x_{n-q}y_{n-q}z_{n-q}}, \quad z_{n+1} = \frac{x_{n-q}y_{n-q}z_{n-q}}{x_{n-r}y_{n-r}z_{n-r}},$$

$n = 0, 1, ...,$ where the initial conditions are nonzero real numbers.

In 2013, Özkan and Kurbanli [3] studied the periodic solutions of the system of rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n},$$

$$z_{n+1} = \frac{x_{n-2} + y_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers.

2 Results

Theorem 2.1. *Let β be a nonzero real number, and let $y_0 = a$, $y_{-1} = b$, $y_{-2} = c$, $x_0 = d$, $x_{-1} = e$, $x_{-2} = f$ be arbitrary real numbers such that $fbd \neq -\beta \neq cea$. Let $\{x_n, y_n, z_n\}$ be a solution of the system*

$$x_{n+1} = \frac{\beta y_{n-2}}{-\beta - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{\beta x_{n-2}}{-\beta - x_{n-2}y_{n-1}x_n},$$

$$z_{n+1} = \frac{\beta x_{n-2} + \beta y_{n-2}}{-\beta - x_{n-2}y_{n-1}x_n}, \quad n = 0, 1, ...$$

Then all six-period solutions of the system are as follows:

$$x_{6n+1} = -\frac{\beta c}{cea + \beta}, \quad y_{6n+1} = -\frac{\beta f}{fbd + \beta}, \quad z_{6n+1} = -\frac{\beta f + \beta c}{fbd + \beta},$$

$$x_{6n+2} = -\frac{b(fbd + \beta)}{\beta}, \quad y_{6n+2} = -\frac{e(cea + \beta)}{\beta}, \quad z_{6n+2} = -\frac{(e + b)(cea + \beta)}{\beta},$$

$$x_{6n+3} = -\frac{\beta a}{cea + \beta}, \quad y_{6n+3} = -\frac{\beta d}{fbd + \beta}, \quad z_{6n+3} = -\frac{\beta d + \beta a}{fbd + \beta},$$

$$x_{6n+4} = f, \quad y_{6n+4} = c, \quad z_{6n+4} = \frac{c(fbd + \beta) + f(cea + \beta)}{fbd + \beta},$$

$$x_{6n+5} = e, \quad y_{6n+5} = b, \quad z_{6n+5} = \frac{b(fbd + \beta) + e(cea + \beta)}{fbd + \beta},$$

$$x_{6n+6} = d, \quad y_{6n+6} = a, \quad z_{6n+6} = \frac{a(fbd + \beta) + d(cea + \beta)}{fbd + \beta},$$

where $n = 0, 1, ...$

Proof. We note that $x_1 = -\dfrac{\beta c}{cea+\beta}$, $y_1 = -\dfrac{\beta f}{fbd+\beta}$, $z_1 = -\dfrac{\beta f+\beta c}{fbd+\beta}$,

$$x_2 = \frac{\beta y_{-1}}{-\beta - y_{-1}x_0y_1} = \frac{\beta b}{-\beta - bd\left(-\dfrac{\beta f}{fbd+\beta}\right)} = -\frac{b(fbd+\beta)}{\beta},$$

$$y_2 = \frac{\beta x_{-1}}{-\beta - x_{-1}y_0x_1} = \frac{\beta e}{-\beta - ea\left(-\dfrac{\beta c}{cea+\beta}\right)} = -\frac{e(cea+\beta)}{\beta},$$

$$z_2 = \frac{\beta x_{-1}+\beta y_{-1}}{-\beta - x_{-1}y_0x_1} = \frac{\beta e+\beta b}{-\beta - ea\left(-\dfrac{\beta c}{cea+\beta}\right)} = -\frac{(e+b)(cea+\beta)}{\beta},$$

$$x_3 = \frac{\beta y_0}{-\beta - y_0x_1y_2} = \frac{\beta a}{-\beta - a\left(-\dfrac{\beta c}{cea+\beta}\right)\left(-\dfrac{e(cea+\beta)}{\beta}\right)} = -\frac{\beta a}{cea+\beta},$$

$$y_3 = \frac{\beta x_0}{-\beta - x_0y_1x_2} = \frac{\beta d}{-\beta - d\left(-\dfrac{\beta f}{fbd+\beta}\right)\left(-\dfrac{b(fbd+\beta)}{\beta}\right)} = -\frac{\beta d}{fbd+\beta},$$

$$z_3 = \frac{\beta x_0+\beta y_0}{-\beta - x_0y_1x_2} = \frac{\beta d+\beta a}{-\beta - d\left(-\dfrac{\beta f}{fbd+\beta}\right)\left(-\dfrac{b(fbd+\beta)}{\beta}\right)} = -\frac{\beta d+\beta a}{fbd+\beta},$$

$$x_4 = \frac{\beta y_1}{-\beta - y_1x_2y_3} = \frac{\beta\left(-\dfrac{\beta f}{fbd+\beta}\right)}{-\beta - \left(-\dfrac{\beta f}{fbd+\beta}\right)\left(-\dfrac{b(fbd+\beta)}{\beta}\right)\left(-\dfrac{\beta d}{fbd+\beta}\right)} = f,$$

$$y_4 = \frac{\beta x_1}{-\beta - x_1y_2x_3} = \frac{\beta\left(-\dfrac{\beta c}{cea+\beta}\right)}{-\beta - \left(-\dfrac{\beta c}{cea+\beta}\right)\left(-\dfrac{e(cea+\beta)}{\beta}\right)\left(-\dfrac{\beta a}{cea+\beta}\right)} = c,$$

$$\begin{aligned} z_4 &= \frac{\beta x_1+\beta y_1}{-\beta - x_1y_2x_3} \\ &= \frac{\beta\left(-\dfrac{\beta c}{cea+\beta}\right)+\beta\left(-\dfrac{\beta f}{fbd+\beta}\right)}{-\beta - \left(-\dfrac{\beta c}{cea+\beta}\right)\left(-\dfrac{e(cea+\beta)}{\beta}\right)\left(-\dfrac{\beta a}{cea+\beta}\right)} \\ &= \frac{c(fbd+\beta)+f(cea+\beta)}{fbd+\beta}, \end{aligned}$$

$$x_5 = \frac{\beta y_2}{-\beta - y_2x_3y_4} = \frac{\beta\left(-\frac{e(cea+\beta)}{\beta}\right)}{-\beta - \left(-\frac{e(cea+\beta)}{\beta}\right)\left(-\frac{\beta a}{cea+\beta}\right)c} = e,$$

$$y_5 = \frac{\beta x_2}{-\beta - x_2y_3x_4} = \frac{\beta\left(-\frac{b(fbd+\beta)}{\beta}\right)}{-\beta - \left(-\frac{b(fbd+\beta)}{\beta}\right)\left(-\frac{\beta d}{fbd+\beta}\right)f} = b,$$

$$\begin{aligned} z_5 &= \frac{\beta x_2 + \beta y_2}{-\beta - x_2y_3x_4} \\ &= \frac{\beta\left(-\frac{b(fbd+\beta)}{\beta}\right) + \beta\left(-\frac{e(cea+\beta)}{\beta}\right)}{-\beta - \left(-\frac{b(fbd+\beta)}{\beta}\right)\left(-\frac{\beta d}{fbd+\beta}\right)f} \\ &= \frac{b(fbd+\beta) + e(cea+\beta)}{fbd+\beta}, \end{aligned}$$

$$x_6 = \frac{\beta y_3}{-\beta - y_3x_4y_5} = \frac{\beta\left(-\frac{\beta d}{fbd+\beta}\right)}{-\beta - \left(-\frac{\beta d}{fbd+\beta}\right)fb} = d,$$

$$y_6 = \frac{\beta x_3}{-\beta - x_3y_4x_5} = \frac{\beta\left(-\frac{\beta a}{cea+\beta}\right)}{-\beta - \left(-\frac{\beta a}{cea+\beta}\right)ce} = a,$$

$$\begin{aligned} z_6 &= \frac{\beta x_3 + \beta y_3}{-\beta - x_3y_4x_5} \\ &= \frac{\beta\left(-\frac{\beta a}{cea+\beta}\right) + \beta\left(-\frac{\beta d}{fbd+\beta}\right)}{-\beta - \left(-\frac{\beta a}{cea+\beta}\right)ce} \\ &= \frac{a(fbd+\beta) + d(cea+\beta)}{fbd+\beta}. \end{aligned}$$

Then we obtain that

$$x_7 = \frac{\beta y_4}{-\beta - y_4 x_5 y_6} = -\frac{\beta c}{cea + \beta} = x_1,$$
$$y_7 = \frac{\beta x_4}{-\beta - x_4 y_5 x_6} = -\frac{\beta f}{fbd + \beta} = y_1,$$
$$z_7 = \frac{\beta x_4 + \beta y_4}{-\beta - x_4 y_5 x_6} = -\frac{\beta f + \beta c}{fbd + \beta} = z_1,$$

$$x_8 = \frac{\beta y_5}{-\beta - y_5 x_6 y_7} = \frac{\beta b}{-\beta - bd\left(-\frac{\beta f}{fbd + \beta}\right)} = -\frac{b(fbd + \beta)}{\beta} = x_2,$$
$$y_8 = \frac{\beta x_5}{-\beta - x_5 y_6 x_7} = \frac{\beta e}{-\beta - ea\left(-\frac{\beta c}{cea + \beta}\right)} = -\frac{e(cea + \beta)}{\beta} = y_2,$$
$$z_8 = \frac{\beta x_5 + \beta y_5}{-\beta - x_5 y_6 x_7} = \frac{\beta e + \beta b}{-\beta - ea\left(-\frac{\beta c}{cea + \beta}\right)} = -\frac{(e + b)(cea + \beta)}{\beta} = z_2,$$

$$x_9 = \frac{\beta y_6}{-\beta - y_6 x_7 y_8} = \frac{\beta a}{-\beta - a\left(-\frac{\beta c}{cea + \beta}\right)\left(-\frac{e(cea + \beta)}{\beta}\right)} = -\frac{\beta a}{cea + \beta} = x_3,$$
$$y_9 = \frac{\beta x_6}{-\beta - x_6 y_7 x_8} = \frac{\beta d}{-\beta - d\left(-\frac{\beta f}{fbd + \beta}\right)\left(-\frac{b(fbd + \beta)}{\beta}\right)} = -\frac{\beta d}{fbd + \beta} = y_3,$$
$$z_9 = \frac{\beta x_6 + \beta y_6}{-\beta - x_6 y_7 x_8} = \frac{\beta d + \beta a}{-\beta - d\left(-\frac{\beta f}{fbd + \beta}\right)\left(-\frac{b(fbd + \beta)}{\beta}\right)} = -\frac{\beta d + \beta a}{fbd + \beta} = z_3,$$

$$x_{10} = \frac{\beta\left(-\frac{\beta f}{fbd + \beta}\right)}{-\beta - \left(-\frac{\beta f}{fbd + \beta}\right)\left(-\frac{b(fbd + \beta)}{\beta}\right)\left(-\frac{\beta d}{fbd + \beta}\right)} = f = x_4,$$

$$y_{10} = \frac{\beta\left(-\frac{\beta c}{cea + \beta}\right)}{-\beta - \left(-\frac{\beta c}{cea + \beta}\right)\left(-\frac{e(cea + \beta)}{\beta}\right)\left(-\frac{\beta a}{cea + \beta}\right)} = c = y_4,$$

$$\begin{aligned} z_{10} &= \frac{\beta x_7 + \beta y_7}{-\beta - x_7 y_8 x_9} \\ &= \frac{\beta\left(-\frac{\beta c}{cea+\beta}\right) + \beta\left(-\frac{\beta f}{fbd+\beta}\right)}{-\beta - \left(-\frac{\beta c}{cea+\beta}\right)\left(-\frac{e(cea+\beta)}{\beta}\right)\left(-\frac{\beta a}{cea+\beta}\right)} \\ &= \frac{c(fbd+\beta) + f(cea+\beta)}{fbd+\beta} \\ &= z_4, \end{aligned}$$

$$x_{11} = \frac{\beta y_8}{-\beta - y_8 x_9 y_{10}} = \frac{\beta\left(-\frac{e(cea+\beta)}{\beta}\right)}{-\beta - \left(-\frac{e(cea+\beta)}{\beta}\right)\left(-\frac{\beta a}{cea+\beta}\right)c} = e = x_5,$$

$$y_{11} = \frac{\beta x_8}{-\beta - x_8 y_9 x_{10}} = \frac{\beta\left(-\frac{b(fbd+\beta)}{\beta}\right)}{-\beta - \left(-\frac{b(fbd+\beta)}{\beta}\right)\left(-\frac{\beta d}{fbd+\beta}\right)f} = b = y_5,$$

$$\begin{aligned} z_{11} &= \frac{\beta x_8 + \beta y_8}{-\beta - x_8 y_9 x_{10}} \\ &= \frac{\beta\left(-\frac{b(fbd+\beta)}{\beta}\right) + \beta\left(-\frac{e(cea+\beta)}{\beta}\right)}{-\beta - \left(-\frac{b(fbd+\beta)}{\beta}\right)\left(-\frac{\beta d}{fbd+\beta}\right)f} \\ &= \frac{b(fbd+\beta) + e(cea+\beta)}{fbd+\beta} \\ &= z_5, \end{aligned}$$

$$x_{12} = \frac{\beta y_9}{-\beta - y_9 x_{10} y_{11}} = \frac{\beta\left(-\frac{\beta d}{fbd+\beta}\right)}{-\beta - \left(-\frac{\beta d}{fbd+\beta}\right)fb} = d = x_6,$$

$$y_{12} = \frac{\beta x_9}{-\beta - x_9 y_{10} x_{11}} = \frac{\beta\left(-\frac{\beta a}{cea+\beta}\right)}{-\beta - \left(-\frac{\beta a}{cea+\beta}\right)ce} = a = y_6,$$

$$\begin{aligned} z_{12} &= \frac{\beta x_9 + \beta y_9}{-\beta - x_9 y_{10} x_{11}} \\ &= \frac{\beta\left(-\dfrac{\beta a}{cea+\beta}\right) + \beta\left(-\dfrac{\beta d}{fbd+\beta}\right)}{-\beta - \left(-\dfrac{\beta a}{cea+\beta}\right) ce} \\ &= \frac{a(fbd+\beta) + d(cea+\beta)}{fbd+\beta} \\ &= z_6. \end{aligned}$$

Next, we let $m \in \mathbb{N}$. Suppose that $x_{6m+1} = x_1$, $y_{6m+1} = y_1$, $z_{6m+1} = z_1$, $x_{6m+2} = x_2$, $y_{6m+2} = y_2$, $z_{6m+2} = z_2$, $x_{6m+3} = x_3$, $y_{6m+3} = y_3$, $z_{6m+3} = z_3$, $x_{6m+4} = x_4$, $y_{6m+4} = y_4$, $z_{6m+4} = z_4$, $x_{6m+5} = x_5$, $y_{6m+5} = y_5$, $z_{6m+5} = z_5$, $x_{6m+6} = x_6$, $y_{6m+6} = y_6$ and $z_{6m+6} = z_6$. Then

$$\begin{aligned} x_{6m+7} &= \frac{\beta y_{6m+4}}{-\beta - y_{6m+4} x_{6m+5} y_{6m+6}} = \frac{\beta y_4}{-\beta - y_4 x_5 y_6} = x_7 = x_1, \\ y_{6m+7} &= \frac{\beta x_{6m+4}}{-\beta - x_{6m+4} y_{6m+5} x_{6m+6}} = \frac{\beta x_4}{-\beta - x_4 y_5 x_6} = y_7 = y_1, \\ z_{6m+7} &= \frac{\beta x_{6m+4} + \beta y_{6m+4}}{-\beta - x_{6m+4} y_{6m+5} x_{6m+6}} = \frac{\beta x_4 + \beta y_4}{-\beta - x_4 y_5 x_6} = z_7 = z_1, \end{aligned}$$

$$\begin{aligned} x_{6m+8} &= \frac{\beta y_{6m+5}}{-\beta - y_{6m+5} x_{6m+6} y_{6m+7}} = \frac{\beta y_5}{-\beta - y_5 x_6 y_7} = x_8 = x_2, \\ y_{6m+8} &= \frac{\beta x_{6m+5}}{-\beta - x_{6m+5} y_{6m+6} x_{6m+7}} = \frac{\beta x_5}{-\beta - x_5 y_6 x_7} = y_8 = y_2, \\ z_{6m+8} &= \frac{\beta x_{6m+5} + \beta y_{6m+5}}{-\beta - x_{6m+5} y_{6m+6} x_{6m+7}} = \frac{\beta x_5 + \beta y_5}{-\beta - x_5 y_6 x_7} = z_8 = z_2, \end{aligned}$$

$$\begin{aligned} x_{6m+9} &= \frac{\beta y_{6m+6}}{-\beta - y_{6m+6} x_{6m+7} y_{6m+8}} = \frac{\beta y_6}{-\beta - y_6 x_7 y_8} = x_9 = x_3, \\ y_{6m+9} &= \frac{\beta x_{6m+6}}{-\beta - x_{6m+6} y_{6m+7} x_{6m+8}} = \frac{\beta x_6}{-\beta - x_6 y_7 x_8} = y_9 = y_3, \\ z_{6m+9} &= \frac{\beta x_{6m+6} + \beta y_{6m+6}}{-\beta - x_{6m+6} y_{6m+7} x_{6m+8}} = \frac{\beta x_6 + \beta y_6}{-\beta - x_6 y_7 x_8} = z_9 = z_3, \end{aligned}$$

$$\begin{aligned} x_{6m+10} &= \frac{\beta y_{6m+7}}{-\beta - y_{6m+7} x_{6m+8} y_{6m+9}} = \frac{\beta y_7}{-\beta - y_7 x_8 y_9} = x_{10} = x_4, \\ y_{6m+10} &= \frac{\beta x_{6m+7}}{-\beta - x_{6m+7} y_{6m+8} x_{6m+9}} = \frac{\beta x_7}{-\beta - x_7 y_8 x_9} = y_{10} = y_4, \\ z_{6m+10} &= \frac{\beta x_{6m+7} + \beta y_{6m+7}}{-\beta - x_{6m+7} y_{6m+8} x_{6m+9}} = \frac{\beta x_7 + \beta y_7}{-\beta - x_7 y_8 x_9} = z_{10} = z_4, \end{aligned}$$

$$x_{6m+11} = \frac{\beta y_{6m+8}}{-\beta - y_{6m+8}x_{6m+9}y_{6m+10}} = \frac{\beta y_8}{-\beta - y_8x_9y_{10}} = x_{11} = x_5,$$
$$y_{6m+11} = \frac{\beta x_{6m+8}}{-\beta - x_{6m+8}y_{6m+9}x_{6m+10}} = \frac{\beta x_8}{-\beta - x_8y_9x_{10}} = y_{11} = y_5,$$
$$z_{6m+11} = \frac{\beta x_{6m+8} + \beta y_{6m+8}}{-\beta - x_{6m+8}y_{6m+9}x_{6m+10}} = \frac{\beta x_8 + \beta y_8}{-\beta - x_8y_9x_{10}} = z_{11} = z_5,$$

$$x_{6m+12} = \frac{\beta y_{6m+9}}{-\beta - y_{6m+9}x_{6m+10}y_{6m+11}} = \frac{\beta y_9}{-\beta - y_9x_{10}y_{11}} = x_{12} = x_6,$$
$$y_{6m+12} = \frac{\beta x_{6m+9}}{-\beta - x_{6m+9}y_{6m+10}x_{6m+11}} = \frac{\beta x_9}{-\beta - x_9y_{10}x_{11}} = y_{12} = y_6,$$
$$z_{6m+12} = \frac{\beta x_{6m+9} + \beta y_{6m+9}}{-\beta - x_{6m+9}y_{6m+10}x_{6m+11}} = \frac{\beta x_9 + \beta y_9}{-\beta - x_9y_{10}x_{11}} = z_{12} = z_6.$$

By the mathematical induction, this proof is completed. □

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