On Some Recurrence Relations of Generalized q-Mittag Leffler Function

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Abstract
In this paper, we investigate the q-difference relation of q-analogue of generalized Mittag Leffler function by using technique of q-calculus and also investigate some properties by using q-derivative.

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1. Introduction
In 1903, the Swedish mathematician Gosta Mittag Leffler [5] introduced the function $E_\alpha(z)$ by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, \text{Re}(\alpha) > 0) \quad (1.1)$$

The generalization of $E_\alpha(z)$ was studied by Wiman [14], who defined the function $E_{\alpha,\beta}(z)$ as below

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0) \quad (1.2)$$

In 1971, Prabhakar [6] introduced the function $E'_{\alpha,\beta}(z), \alpha, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0$ which is defined by

$$E'_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (1.3)$$

where $(\lambda)_n$ is the Pochhammer symbol [7] defined by
Where $N$ being the set of positive integers.

Another generalization of Mittag Leffler function $E^\gamma_{\lambda,\beta}(z)$ was studied by T.O. Salim [9], who define the function $E^\gamma_{\lambda,\beta}(z)$ as follows:

$$E^\gamma_{\lambda,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n}$$

We state below the $q$-analogue of above discussed generalized Mittag – Leffter function $E^\gamma_{\lambda,\beta}(z; q)$ as follows

**Definition 1**: For $\alpha, \beta, \gamma \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(\gamma) > 0$ and $|q| < 1$ the function $E^\gamma_{\lambda,\beta}(z; q)$ is defined as

$$E^\gamma_{\lambda,\beta}(z; q) = \sum_{n=0}^{\infty} \frac{(q\gamma)_n}{(q\delta)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}$$

where $\Gamma_q(\lambda)$ is the $q$-gamma function.

The $q$-analogue of the Pochhammer symbol ($q$-shifted factorial) is defined by

$$(\lambda; q)_n = \prod_{k=0}^{n-1} (1 - \lambda q^k) = \frac{(\lambda; q)_\infty}{(\lambda q^n; q)_\infty}$$

and the $q$-analogue of the power $(a - b)^n$ is

$$(a - b)^0 = 1, (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k)$$

There is following relationship between them:

$$(a - b)^n = a^n \left(\frac{\gamma'}{\gamma}; q\right)_n, \quad (a \neq 0)$$

$$= a^n \frac{(\gamma'/\gamma; q)_n}{(q^n \gamma'/\gamma; q)_n}$$

Also, Predrag M. Rajkovic, et. al. [8], define a $q$-derivative of a function $f(z)$ by

$$D_q f(z) = \frac{f(z) - f(qz)}{z - qz} \quad (z \neq 0)$$
Further, the $\Gamma_q(z)$ satisfies the functional equation,

$$
\Gamma_q(z + 1) = \frac{1 - q^z}{1 - q} \Gamma_q(z)
$$

(1.11)

The detailed account of generalized Mittag-Leffler function can be found in research monographs due to Agrawal [1], Kilbas, et. al. [3], Gupta and Debnath [2], Shukla and Prajapati [11, 12, 13] and Sharma and Jain [10].

In this paper, the motive is to evaluate the recurrence relation and the recurrence relation with q-derivative.

2. Recurrence Relations

**Theorem 1**: If $\alpha, \beta, \gamma \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(\gamma) > 0$ then

$$
E_{a, b}^{\gamma, \delta}(z; q) = E_{a, b}^{\gamma, \delta + 1}(z; q) - \frac{q^{\gamma}}{(1 - q^a)} z E_{a, b+1}^{\gamma, \delta + 1}(z; q) - \frac{q^{\gamma+1}}{(1 - q^a)} z E_{a, b}^{\gamma+1, \delta + 1}(qz; q)
$$

(2.1)

**Proof**: From (1.6), we write

$$
E_{a, b}^{\gamma, \delta}(z; q) = \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n z^n}{(q^\delta; q)_n} \frac{1}{\Gamma_q(\alpha n + \beta)} = \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(1 - q^\gamma)(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}
$$

Since $(1 - q^\gamma) = (1 - q^{\gamma+1}) - q^\gamma (1 - q^n)$, the above equation becomes equal to

$$
\frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{1}{\Gamma_q(\alpha n + \beta)} [1 - q^\gamma (1 - q^n)] z^n
$$

$$
= \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} - \frac{q^\gamma}{1 - q^a} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{1 - q^n}{\Gamma_q(\alpha n + \beta)} z^n
$$

$$
= \sum_{n=0}^{\infty} \frac{(q^{\gamma+1}; q)_n z^n}{(q^\delta; q)_n} \frac{1}{\Gamma_q(\alpha n + \beta)} - \frac{q^\gamma}{1 - q^a} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{1 - q^n}{\Gamma_q(\alpha n + \beta)} z^n
$$

$$
= \sum_{n=0}^{\infty} \frac{(q^{\gamma+1}; q)_n z^n}{(q^\delta; q)_n} \frac{1}{\Gamma_q(\alpha n + \beta)} - \frac{q^\gamma}{1 - q^a} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{1 - q^n}{\Gamma_q(\alpha n + \beta)} z^n
$$

$$
- \frac{q^\gamma}{(1 - q^a)} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_{n-1}}{(q^\delta; q)_n} \frac{(qz)^n}{\Gamma_q(\alpha n + \beta)}
$$
On replacing $n$ by $m+1$ in second and third summation, the RHS of above equation becomes

$$
\sum_{n=0}^{\infty} \frac{(q^{\gamma+1}; q)_n z^n}{\Gamma_q (\alpha n + \beta)} - \frac{q^{\gamma}}{(1 - q^{\delta})} \sum_{m=0}^{\infty} \frac{(q^{\delta+1}; q)_m z^m}{\Gamma_q (\alpha m + (\alpha + \beta))} \frac{q^{\gamma+1}}{(1 - q^{\delta})} \sum_{n=0}^{\infty} \frac{(q^{\gamma+1}; q)_n (q^{\gamma+1}; q)_m (gz)^m}{\Gamma_q (\alpha m + (\alpha + \beta))}
$$

In view of definition (1.6), the above expression becomes

$$
E^{\gamma+1,\delta}_{\alpha,\beta} (z; q) - \frac{q^{\gamma}}{(1 - q^{\delta})} z E^{\gamma+1,\delta+1}_{\alpha,\alpha+\beta} (z; q) - \frac{q^{\gamma+1}}{(1 - q^{\delta})} z E^{\gamma+1,\delta+1}_{\alpha,\beta} (gz; q)
$$

This completes the proof of the result (2.1).

**Theorem 2 :** Let $\alpha, \beta, \gamma, \omega \in \mathbb{C}$, then for any $n = 1, 2, 3, ...$

$$
D_q [z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta} (\omega z^\alpha; q)] = z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta} (\omega z^\alpha; q) \quad (2.2)
$$

where $\text{Re}(\beta) > n$.

**Proof :** Consider the function

$$
f(z) = z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta} (\omega z^\alpha; q) \quad \text{in (1.10)} \quad \text{and applying the definition (1.6)}
$$

$$
D_q [z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta} (\omega z^\alpha; q)] \quad \text{becomes}
$$

$$
\sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n (1 - q^{\alpha n + \beta - 1}) \omega z^{\alpha n + \beta - 2}}{(1 - q) \Gamma_q (\alpha n + \beta)}
$$

According to the functional equation (1.11) the above expression becomes

$$
\sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n \omega z^{\alpha n + \beta - 2}}{(1 - q) \Gamma_q (\alpha n + \beta - 1)}
$$

which equals $z^{\beta-2} E^{\gamma,\delta}_{\alpha,\beta-1} (\omega z^\alpha; q)$

Finally, we obtain

$$
D_q [z^{\beta-1} E^{\gamma,\delta}_{\alpha,\beta} (\omega z^\alpha; q)] = z^{\beta-2} E^{\gamma,\delta}_{\alpha,\beta-1} (\omega z^\alpha; q)
$$

Iterating this result, upto $n-1$ times, we obtain the required formula.
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References


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