ON SOME PROPERTIES OF GENERALIZED q-MITTAG LEFFLER FUNCTION

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Abstract
In the present paper, we make an attempt to introduce q-analogue of
generalized Mittag Leffler function $E_{\alpha,\beta}(z;q)$ and its q-recurrence relations with
q-derivative. Also, we present q-fractional operators and properties of $E_{\alpha,\beta}^\gamma(z;q)$
by using fractional q-calculus.

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derivative.

1. Introduction
In 1903, the Swedish mathematician Gosta Mittag Leffler [5] introduced
the function $E_\alpha(z)$ by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, \ \text{Re}(\alpha) > 0) \quad (1.1)$$

The generalization of $E_\alpha(z)$ was studied by Wiman [12], who defined the
function $E_{\alpha,\beta}(z)$ as below
\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0) \] (1.2)

In 1971, Prabhakar [6] introduced the function \( E_{\alpha,\beta}^\gamma(z) \), \( \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \) which is defined by

\[ E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \] (1.3)

where \((\lambda)_n\) is the Pochhammer symbol [7] defined by

\[ (\lambda)_n = \begin{cases} 1, & n = 0, \lambda \neq 0 \\ \lambda(\lambda+1) \cdots (\lambda+n-1), & n \in \mathbb{N}, \lambda \in \mathbb{C} \end{cases} \] (1.4)

\( \mathbb{N} \) being the set of positive integers.

In the sequel to this study, we define the q-analogue of generalized Mittag – Leffler function \( E_{\alpha,\beta}^\gamma(z; q) \) as follows

**Definition 1**: For \( \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0 \) and \(|q| < 1\) the function \( E_{\alpha,\beta}^\gamma(z; q) \) is defined as

\[ E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(q; q)_n \Gamma_q(\alpha n + \beta)} \frac{z^n}{n!} \] (1.5)

where \( \Gamma_q(\lambda) \) is the q-gamma function.

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by

\[ (\lambda; q)_n = \prod_{k=0}^{n-1} (1 - \lambda q^k) = \frac{(\lambda; q)_\infty}{(\lambda q^k; q)_\infty} \] (1.6)

and the q-analogue of the power \((a - b)^n\) is

\[ (a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k) \] (1.7)

There is following relationship between them:

\[ (a - b)^n = a^n (\frac{\gamma_q}{a}; q)_n, \quad (a \neq 0) \]
\[ = a^n \frac{(q/\alpha, q)_\infty}{(q^{n/\gamma}, q)_\infty} \] (1.8)

Also, Predrag M. Rajkovic, et. al. [8], define a q-derivative of a function \( f(z) \) by

\[ D_q f(z) = \frac{f(z) - f(qz)}{z - qz} \quad (z \neq 0) \] (1.9)

Further, the \( \Gamma_q(z) \) satisfies the functional equation,

\[ \Gamma_q(z + 1) = \frac{1 - q^{z+1}}{1 - q} \Gamma_q(z) \] (1.10)

Again, the q-analogue of the beta function is defined by

\[ B_q(x, y) = \frac{x^{-1} (tq^{-1}; q)_\infty}{(tq^{-1}; q)_\infty} d_q(t) \] (1.11)

The relation between q-beta function and q-gamma function is

\[ B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x + y)} , \text{ (Re}(x) > 0, \text{ Re}(y) > 0) \]

(1.12)

The detailed account of generalized Mittag-Leffler function can be found in research monographs due to Agrawal [1], Kilbas, et. al. [3], Gupta and Debnath [2], and Shukla and Prajapati [9, 10, 11].

In this paper, the motive is to evaluate the recurrence relation with q-derivative and in the last section, properties of \( E_{\alpha,\beta}^\gamma(z; q) \) by using fractional q-calculus.

2. Recurrence Relations

**Theorem 1**: If \( \alpha, \beta, \gamma \in \mathbb{C}, \text{ Re}(\alpha) > 0, \text{ Re}(\beta) > 0, \text{ Re}(\gamma) > 0 \) then

\[ E_{\alpha,\beta}^\gamma(z; q) = E_{\alpha,\beta}^{\gamma+1}(z; q) - q^\gamma E_{\alpha,\beta+1}(z; q) \] (2.1)

**Proof**: From (1.5), we write

\[ E_{\alpha,\beta}^\gamma(z; q) = \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} = \frac{1}{\Gamma_q(\beta)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} \]
\[ = \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(1-q^z)(q^{\gamma+1}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} \]

Since \((1-q^z) = (1-q^{\gamma+n}) - q^\gamma (1-q^n)\) then, the above equation becomes equal to

\[
= \frac{1}{\Gamma_q(\beta)} + \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} - \frac{1}{\Gamma_q(\beta)} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}
\]

On replacing n by m+1 in second summation, the RHS of above equation becomes

\[
= \sum_{n=0}^{\infty} \frac{(q^{\gamma+1}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)} - \frac{1}{\Gamma_q(\beta)} \sum_{n=1}^{\infty} \frac{(q^{\gamma+1}; q)_n}{(q; q)_n} \frac{z^n}{\Gamma_q(\alpha n + \beta)}
\]

In view of definition (1.5), the above expression becomes

\[E_{\alpha,\beta}^{\gamma+1}(z; q) - \gamma E_{\alpha,\alpha+\beta}^{\gamma+1}(z; q)\]

This completes the proof of the result (2.1).

**Theorem 2 :** Let \(\alpha, \beta, \gamma, \omega \in \mathbb{C}\), then for any \(n = 1, 2, 3, \ldots\)

\[D_q^n[z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\alpha \omega^n; q)] = z^{\beta-n-1}E_{\alpha,\beta-n}^{\gamma}(\alpha \omega^n; q)\] (2.2)

where \(\text{Re}(\beta) > n\).

**Proof :** Consider the function

\[f(z) = z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\alpha \omega^n; q)\] in (1.9) and applying the definition (1.5)

\[D_q^n[z^{\beta-1}E_{\alpha,\beta}^{\gamma}(\alpha \omega^n; q)]\] becomes

\[
\sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n}{(q; q)_n} \frac{(1-q^m \beta-1)}{(1-q)} \omega^n z^{\alpha n + \beta-2} \frac{1}{\Gamma_q(\alpha n + \beta)}
\]

According to the functional equation (1.10) the above expression becomes

\[
\sum_{n=0}^{\infty} \frac{(q^{\gamma}; q)_n}{(q; q)_n} \omega^n z^{\alpha n + \beta-2} \frac{1}{\Gamma_q(\alpha n + \beta - 1)}
\]
which equals $z^{\beta-2}E_{\alpha,\beta-1}^\gamma (\alpha \xi^n; q)$

Finally, we obtain

$$D_q[z^{\beta-1}E_{\alpha,\beta}^\gamma (\alpha \xi^n; q)] = z^{\beta-2}E_{\alpha,\beta-1}^\gamma (\alpha \xi^n; q)$$

Iterating this result, up to $n-1$ times, we obtain the required formula.

### 3. Fractional q-Calculus

We define the fractional q-integral of operator and the fractional q-derivative of Miller and Ross [4] type by

**Definition 2:** The fractional q-integral operator of order $\nu$ defined as for $\text{Re}(\nu) > 0$

$$I_q^\nu f(t) = \frac{1}{\Gamma_q(\nu)} \int_0^t (t-q\xi)^{(\nu-1)} f(\xi) d_q\xi$$  \hspace{1cm} (3.1)

**Definition 3:** The fractional q-differential operator of order $\mu$ defined as

$$D_q^\mu f(t) = \frac{1}{\Gamma_q(k)} \{I_q^{k-\mu} f(t)\}$$  \hspace{1cm} (3.2)

where $\text{Re}(\mu) > 0$ and if $k$ is the smallest integer with the property that $k \geq \text{Re}(\mu)$.

**Theorem 3:** Let $\gamma \in \mathbb{C}$, $\text{Re}(\gamma) > 0$ and $c$ is any arbitrary constant, then

$$I_q^\nu E_{\nu,\nu+1}^\gamma (ct; q) = t^\nu E_{\nu,\nu+1}^\gamma (ct; q)$$  \hspace{1cm} (3.3)

**Proof:** Consider the function $f(t) = E_{\nu,\nu+1}^\gamma (ct; q)$ in (3.1) and applying the definition (1.5), the LHS of above expression becomes.

$$\frac{1}{\Gamma_q(\nu)} \int_0^t (t-q\xi)^{(\nu-1)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n \Gamma_q(n+1)} d_q\xi$$

Now, using relation (1.8) above expression reduce to

$$\frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n} c^n \int_0^t \frac{t^{-n}(q^\gamma t; q)}{q^{\gamma} t^{\nu} (q; q)_n} d_q\xi$$

On simplification, we have

$$\frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n}{(q; q)_n} c^n \int_0^t \left(\frac{\xi^n}{t^n}\right)_{q^\gamma} \frac{q^\gamma t; q}{q^\gamma t^{\nu} (q; q)_n} d_q\xi$$
substituting $\xi = xt$, which yields

$$
= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n c^n}{(q; q)_n} t^{\nu n} \int_0^\infty x^n \frac{(q x; q)_\infty}{(q^\gamma x; q)_\infty} d_q x
$$

Using the definition of beta function (1.11), the above expression becomes

$$
= \frac{1}{\Gamma_q(\nu)} \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n c^n}{(q; q)_n} t^{\nu n} B_q(n+1, \nu)
$$

Also, using the relation (1.12) and on simplification, the RHS of above equation reduce to

$$
t^\nu \sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n (ct)^n}{(q; q)_n} \frac{1}{\Gamma_q(\nu + n + 1)} = t^\nu E_{1,\nu+1}(ct; q)
$$

This completes the proof of the result (3.3).

**Theorem 4**: Let $\gamma \in \mathbb{C}$, $\text{Re}(\gamma) > 0$ and $c$ is any arbitrary constant, then

$$
D_q^\nu E_{(1)}(ct; q) = t^{-\mu} E_{1,\nu+1}(ct; q)
$$

(3.4)

**Proof**: Consider the function $f(t) = E_{(1)}(ct; q)$ in (3.2)

The LHS of above expression reduce to

$$
D_q^k \left\{ t^{k-\mu} E_{(1)}(ct; q) \right\}
$$

Now using theorem (3) it becomes

$$
D_q^k \left\{ t^{k-\mu} E_{1,k-\mu+1}(ct; q) \right\}
$$

Using definition (1.5) and on simplification we have

$$
\sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n c^n}{(q; q)_n} D_q^k \left\{ \frac{t^{k-\mu n}}{\Gamma_q(\nu + n + 1)} \right\}
$$

On applying (1.9) and (1.10) upto $k$ times the above equation reduce to

$$
\sum_{n=0}^{\infty} \frac{(q^\gamma; q)_n c^n}{(q; q)_n} \frac{t^{-\mu n}}{\Gamma_q(n+1-\mu)}
$$

It can be written as $t^{-\mu} E_{1,\nu+1}(ct; q)$

This completes the proof of the result (3.4).
References


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