On \( g\alpha r \)-Connectedness and \( g\alpha r \)-Compactness in Topological Spaces

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Abstract

In this paper, the authors introduce a new type of connected spaces called generalized \( \alpha \) regular-connected spaces (briefly \( g\alpha r \)-connected spaces) in topological spaces. The notion of generalized \( \alpha \) regular-compact spaces is also introduced (briefly \( g\alpha r \)-compact spaces) in topological spaces. Some characterizations and several properties concerning \( g\alpha r \)-connected spaces and \( g\alpha r \)-compact spaces are obtained.

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1 Introduction

Topological spaces are mathematical structures that allow the formal definitions of concepts such as connectedness, compactness, interior and closure. In 1974, Das [4] defined the concept of semi-connectedness in topology and investigated its properties. Compactness is one of the most important, useful and fundamental concepts in topology. In 1981, Dorsett [6] introduced and studied the concept of semi-compact spaces. In 1990, Ganster [7] defined and investigated semi-Lindelof spaces. Since then, Hanna and Dorsett
[10], Ganster and Mohammad S. Sarsak [8] investigated the properties of semi-
compact spaces. The notion of connectedness and compactness are useful and
fundamental notions of not only general topology but also of other advanced
branches of mathematics. Ganster and Steiner [9] introduced and studied the
gb-compactness and gb-connectedness in topological spaces. Dontchev and
analyzed rg-compactness and rg-connectedness. Crossely et al [3] introduced
in topological spaces. The aim of this paper is to introduce the concept of
gαr-connected and gαr-compactness in topological spaces.

2 Preliminary Notes

Definition 2.1. A subset $A$ of a topological space $(X,\tau)$, is called sg closed,
if $\text{scl}(A) \subseteq U$. The complement of sg closed set is said to be sg open set. The
family of all sg open sets (respectively semi generalised closed sets) of $(X,\tau)$
is denoted by $SG-O(X,\tau)$[respectively $SG-\text{CL}(X,\tau)$].

Definition 2.2. A subset $A$ of a topological space $(X,\tau)$, is called generalized
$\alpha$ regular-closed set [11] (briefly $g\alpha r$-closed set) if $\alpha\text{cl}(A) \subseteq U$ whenever
$A \subseteq U$ and $U$ is regular open in $X$. The complement of $g\alpha r$-closed set is called
g$\alpha r$-open. The family of all $g\alpha r$-open [respectively $g\alpha r$-closed] sets of $(X,\tau)$
is denoted by $g\alpha r-O(X,\tau)$ [respectively $g\alpha r-\text{CL}(X,\tau)$].

Definition 2.3. A subset $A$ of a topological space $(X,\tau)$ is called b-open
set[1] if $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$. The complement of b-open set is b-
closed sets. The family of all b-open sets (respectively b-closed sets) of $(X,\tau)$
is denoted by $bO(X,\tau)$ (respectively $b\text{CL}(X,\tau)$)

Definition 2.4. The $g\alpha r$-closure of a set $A$, denoted by $g\alpha r-\text{Cl}(A)$[12] is
the intersection of all g$\alpha r$-closed sets containing $A$.

Definition 2.5. The $g\alpha r$-interior of a set $A$, denoted by $g\alpha r-\text{int}(A)$[12] is
the union of all $g\alpha r$-open sets containing $A$.

Definition 2.6. A topological space $X$ is said to be gb-connected [2] if $X$
cannot be expressed as a disjoint of two non-empty gb-open sets in $X$. A sub
set of $X$ is gb-connected if it is gb-connected as a subspace.

Definition 2.7. A subset $A$ of a topological space $(X,\tau)$ is called generalized
$\alpha$ regular-closed set[11] (briefly $g\alpha r$-closed set) if $\alpha\text{cl}(A) \subseteq U$ whenever $A \subseteq U$
and $U$ is regular open in $X$. 

3 Main Results $g\alpha r$-Connectedness

**Definition 3.1.** A topological space $X$ is said to be $g\alpha r$-connected if $X$ cannot be expressed as a disjoint of two non-empty $g\alpha r$-open sets in $X$. A subset of $X$ is $g\alpha r$-connected if it is $g\alpha r$-connected as a subspace.

**Example 3.2.** Let $X = \{a, b, c\}$ and let $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. It is $g\alpha r$-connected.

**Theorem 3.3.** For a topological space $X$, the following are equivalent.

(i) $X$ is $g\alpha r$-connected.

(ii) $X$ and $\varnothing$ are the only subsets of $X$ which are both $g\alpha r$-open and $g\alpha r$-closed.

(iii) Each $g\alpha r$-continuous map of $X$ into a discrete space $Y$ with at least two points is constant map.

**Proof.** (i) $\Rightarrow$ (ii) : Suppose $X$ is $g\alpha r$ - connected. Let $S$ be a proper subset which is both $g\alpha r$ - open and $g\alpha r$ - closed in $X$. Its complement $X - S$ is also $g\alpha r$ - open and $g\alpha r$ - closed. $X = S \cup (X - S)$, a disjoint union of two non-empty $g\alpha r$ - open sets which is contradicts (i). Therefore $S = \varnothing$ or $X$.

(ii) $\Rightarrow$ (i) : Suppose that $X = A \cup B$ where $A$ and $B$ are disjoint non-empty $g\alpha r$ - open subsets of $X$. Then $A$ is both $g\alpha r$ - open and $g\alpha r$ - closed. By assumption $A = \varnothing$ or $X$. Therefore $X$ is $g\alpha r$ - connected.

(ii) $\Rightarrow$ (iii) : Let $f : X \to Y$ be a $g\alpha r$ - continuous map. $X$ is covered by $g\alpha r$ - open and $g\alpha r$ - closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption $f^{-1}(y) = \varnothing$ or $X$ for each $y \in Y$. If $f^{-1}(y) = \varnothing$ for all $y \in (Y)$, then $f$ fails to be a map. Then there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \varnothing$ and hence $f^{-1}(y) = X$. This shows that $f$ is a constant map.

(iii) $\Rightarrow$ (ii) : Let $S$ be both $g\alpha r$ - open and $g\alpha r$ - closed in $X$. Suppose $S \neq \varnothing$. Let $f : X \to Y$ be a $g\alpha r$ - continuous function defined by $f(S) = \{y\}$ and $f(X - S) = \{w\}$ for some distinct points $y$ and $w$ in $Y$. By (iii) $f$ is a constant function. Therefore $S = X$. □

**Theorem 3.4.** Every $g\alpha r$ - connected space is connected.

**Proof.** Let $X$ be $g\alpha r$ - connected. Suppose $X$ is not connected. Then there exists a proper non-empty subset $B$ of $X$ which is both open and closed in $X$. Since every closed set is $g\alpha r$ - closed, $B$ is a proper non-empty subset of $X$ which is both $g\alpha r$ - open and $g\alpha r$ - closed in $X$. Using by Theorem 3.3, $X$ is not $g\alpha r$ - connected. This proves the theorem. □

The converse of the above theorem need not be true as shown in the following example.
Example 3.5. Let $X = \{a, b, c\}$ and let $\tau = \{X, \emptyset, \{b\}, \{c\}, \{a, c\}\}$. $X$ is connected but not gar-connected. Since $\{b\}, \{a, c\}$ are disjoint gar-open sets and $X = \{b\} \cup \{a, c\}$.

Theorem 3.6. If $f : X \to Y$ is a gar-continuous onto and $X$ is gar-connected, then $Y$ is connected.

Proof. Suppose that $Y$ is not connected. Let $Y = A \cup B$ where $A$ and $B$ are disjoint non-empty open set in $Y$. Since $f$ is gar-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gar-open sets in $X$. This contradicts the fact that $X$ is gar-connected. Hence $Y$ is connected.

Theorem 3.7. If $f : X \to Y$ is a gar-irresolut and $X$ is gar-connected, then $Y$ is gar-connected.

Proof. Suppose that $Y$ is not gar-connected. Let $Y = A \cup B$ where $A$ and $B$ are disjoint non-empty gar-open sets in $Y$. Since $f$ is gar-irresolut and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gar-open sets in $X$. This contradicts the fact that $X$ is gar-connected. Hence $Y$ is gar-connected.

Definition 3.8. A topological space $X$ is said to be $T_{gar}$-space if every gar-closed set of $X$ is closed subset of $X$.

Theorem 3.9. Suppose that $X$ is $T_{gar}$-space then $X$ is connected if and only if it is gar-connected.

Proof. Suppose that $X$ is connected. Then $X$ cannot be expressed as disjoint union of two non-empty proper subsets of $X$. Suppose $X$ is not a gar-connected space. Let $A$ and $B$ be any two gar-open subsets of $X$ such that $X = A \cup B$, where $A \cap B = \emptyset$ and $A \subset X, B \subset X$. Since $X$ is $T_{gar}$-space and $A, B$ are gar-open, $A, B$ are open subsets of $X$, which contradicts that $X$ is connected. Therefore $X$ is gar-connected. Conversely, every open set is gar-open. Therefore every gar-connected space is connected.

Theorem 3.10. If the gar-open sets $C$ and $D$ form a separation of $X$ and if $Y$ is gar-connected subspace of $X$, then $Y$ lies entirely within $C$ or $D$.

Proof. Since $C$ and $D$ are both gar-open in $X$, the sets $C \cap Y$ and $D \cap Y$ are gar-open in $Y$. These two sets are disjoint and their union is $Y$. If they were both non-empty, they would constitute a separation of $Y$. Therefore, one of them is empty. Hence $Y$ must lie entirely $C$ or $D$. 


Theorem 3.11. Let $A$ be a $gαr$-connected subspace of $X$. If $A \subset B \subset gαr - cl(A)$ then $B$ is also $gαr$-connected.

Proof. Let $A$ be $gαr$-connected and let $A \subset B \subset gαr - cl(A)$. Suppose that $B = C \cup D$ is a separation of $B$ by $gαr$-open sets. By using Theorem 3.10, $A$ must lie entirely in $C$ or $D$. Suppose that $A \subset C$, then $gαr - cl(A) \subset gαr - cl(B)$. Since $gαr - cl(C)$ and $D$ are disjoint, $B$ cannot intersect $D$. This contradicts the fact that $C$ is non empty subset of $B$. So $D = \varnothing$ which implies $B$ is $gαr$-connected. □

Theorem 3.12. A contra $gαr$-continuous image of an $gαr$-connected space is connected.

Proof. Let $f : X \rightarrow Y$ is a contra $gαr$-continuous function from $gαr$-connected space $X$ on to a space $Y$. Assume that $Y$ is disconnected. Then $Y = A \cup B$, where $A$ and $B$ are non empty clopen sets in $Y$ with $A \cap B = \varnothing$. Since $f$ is contra $gαr$-continuous, we have $f^{-1}(A)$ and $f^{-1}(B)$ are non empty $gαr$-open sets in $X$ with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\varnothing) = \varnothing$. This shows that $X$ is not $gαr$-connected, which is a contradiction. This proves the theorem. □

4 Main Results $gαr$-Compactness

Definition 4.1. A collection $\{A_\alpha : \alpha \in \Lambda\}$ of $gαr$-open sets in a topological space $X$ is called a $gαr$-open cover of a subset $B$ of $X$ if $B \subset \bigcup\{A_\alpha : \alpha \in \Lambda\}$ holds.

Definition 4.2. A topological space $X$ is $gαr$-compact if every $gαr$-open cover of $X$ has a finite sub-cover.

Definition 4.3. A subset $B$ of a topological space $X$ is said to be $gαr$-compact relative to $X$, if for every collection $\{A_\alpha : \alpha \in \Lambda\}$ of $gαr$-open subsets of $X$ such that $B \subset \bigcup\{A_\alpha : \alpha \in \Lambda\}$ there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $B \subset \bigcup\{A_\alpha : \alpha \in \Lambda_0\}$.

Definition 4.4. A subset $B$ of a topological space $X$ is said to be $gαr$-compact if $B$ is $gαr$-compact as a subspace of $X$.

Theorem 4.5. Every $gαr$-closed subset of $gαr$-compact space is $gαr$-compact relative to $X$.

Proof. Let $A$ be $gαr$-closed subset of a $gαr$-compact space $X$. Then $A^c$ is $gαr$-open in $X$. Let $M = \{G_\alpha : \alpha \in \Lambda\}$ be a cover of $A$ by $gαr$-open sets in $X$. Then $M^* = M \cup A^c$ is a $gαr$-open cover of $X$. Since $X$ is $gαr$-compact, $M^*$ is reducible to a finite sub cover of $X$, say $X = G_{\alpha_1} \cup$
\[ G_{\alpha_2} \cup G_{\alpha_3} \cup \ldots \cup G_{\alpha_m} \cup A^c, G_{\alpha_k} \in M. \] But \( A \) and \( A^c \) are disjoint. Hence \( A \subset G_{\alpha_1} \cup G_{\alpha_2} \cup G_{\alpha_3} \cup \ldots \cup G_{\alpha_m} G_{\alpha_k} \in M, \) this implies that any \( g\alpha \) open cover \( M \) of \( A \) contains a finite sub-cover. Therefore \( A \) is \( gb \) - compact relative to \( X \). That is, every \( g\alpha \) - closed subset of a \( g\alpha \) - compact space \( X \) is \( g\alpha \) - compact.

**Definition 4.6.** A function \( f : X \rightarrow Y \) is said to be \( g\alpha \) - continuous if \( f^{-1}(V) \) is \( g\alpha \) - closed in \( X \) for every closed set \( V \) of \( Y \).

**Theorem 4.7.** A \( g\alpha \) - continuous image of a \( g\alpha \) - compact space is compact.

*Proof.* Let \( f : X \rightarrow Y \) be a \( g\alpha \) - continuous map from a \( g\alpha \) - compact space \( X \) onto a topological space \( Y \). Let \( \{ A_\alpha : \alpha \in \Lambda \} \) be an open cover of \( Y \). Then \( \{ f^{-1}(A_i) : i \in \Lambda \} \) is a \( g\alpha \) - open cover of \( X \). Since \( X \) is \( g\alpha \) - compact, it has a finite sub-cover say \( \{ f^{-1}(A_1), f^{-1} : i \in \Lambda(A_2), \ldots, f^{-1}(A_n) \} \). Since \( f \) is onto \( \{ A_1, A_2, \ldots, A_n \} \) is a cover of \( Y \), which is finite. Therefore \( Y \) is compact. \qed

**Definition 4.8.** A function \( f : X \rightarrow Y \) is said to be \( g\alpha \) - irresolute if \( f^{-1}(V) \) is \( g\alpha \) - closed in \( X \) for every \( g\alpha \) - closed set \( V \) of \( Y \).

**Theorem 4.9.** If a map \( f : X \rightarrow Y \) is \( g\alpha \) - irresolute and a subset \( B \) of \( X \) is \( g\alpha \) - compact relative to \( X \), then the image \( f(B) \) is \( g\alpha \) - compact relative to \( Y \).

*Proof.* Let \( \{ A_\alpha : \alpha \in \Lambda \} \) be any collection of \( g\alpha \) - open subsets of \( Y \) such that \( f(B) \subset \bigcup \{ A_\alpha : \alpha \in \Lambda \} \subset \bigcup \). Then \( B \subset \bigcup \{ f^{-1}(A_\alpha) : \alpha \in \Lambda \} \). Since by hypothesis \( B \) is \( g\alpha \) - compact relative to \( X \), there exists a finite subset \( \Lambda_0 \in \Lambda \) such that \( B \subset \bigcup \{ f^{-1}(A_\alpha) : \alpha \in \Lambda_0 \} \). Therefore we have \( f(B) \bigcup \subset \bigcup \{ (A_\alpha) : \alpha \in \Lambda_0 \} \), it shows that \( f(B) \) is \( g\alpha \) - compact relative to \( Y \). \qed

**Theorem 4.10.** A space \( X \) is \( g\alpha \) - compact if and only if each family of \( g\alpha \) - closed subsets of \( X \) with the finite intersection property has a non-empty intersection.

*Proof.* Given a collection \( A \) of subsets of \( X \), let \( C = \{ X - A : A \in A \} \) be the collection of their complements. Then the following statements hold.

(a) \( A \) is a collection of \( g\alpha \) - open sets if and only if \( C \) is a collection of \( g\alpha \) - closed sets.

(b) The collection \( A \) covers \( X \) if and only if the intersection \( \bigcap_{C \in C} C \) of all the elements of \( C \) is empty.
The finite sub collection \( \{A_1, A_2, \ldots, A_n\} \) of \( A \) covers \( X \) if and only if the intersection of the corresponding elements \( C_i = X - A_i \) of \( C \) is empty. The statement (a) is trivial, while the (b) and (c) follow from De Morgan’s law. \( X - (\bigcup_{\alpha \in J} A_\alpha) = \bigcap_{\alpha \in J} (X - A_\alpha) \). The proof of the theorem now proceeds in two steps, taking contra positive of the theorem and then the complement. The statement \( X \) is \( gagr \)-compact is equivalent to: Given any collection \( A \) of \( gagr \)-open subsets of \( X \), if \( A \) covers \( X \), then some finite sub collection of \( A \) covers \( X \). This statement is equivalent to its contra positive, which is the following. Given any collection \( A \) of \( gagr \)-open sets, if no finite sub-collection of \( A \) of covers \( X \), then \( A \) does not cover \( X \). Let \( C \) be as earlier, the collection equivalent to the following: Given any collection \( C \) of \( gagr \)-closed sets, if every finite intersection of elements of \( C \) is not-empty, then the intersection of all the elements of \( C \) is non-empty. This is just the condition of our theorem.

Definition 4.11. A space \( X \) is said to be \( gagr \)-Lindelof space if every cover of \( X \) by \( gagr \)-open sets contains a countable sub cover.

Theorem 4.12. Let \( f : X \rightarrow Y \) be a \( gagr \)-continuous surjection and \( X \) be \( gagr \)-Lindelof, then \( Y \) is \( gagr \)-Lindelof Space.

Proof. Let \( f : X \rightarrow Y \) be a \( gagr \)-continuous surjection and \( X \) be \( gagr \)-Lindelof. Let \( \{V_\alpha\} \) be an open cover for \( Y \). Then \( \{f^{-1}(V_\alpha)\} \) is a cover of \( X \) by \( gagr \)-open sets. Since \( X \) is \( gagr \)-Lindelof, \( \{f^{-1}(V_\alpha)\} \) contains a countable sub cover, namely \( \{f^{-1}(V_{\alpha n})\} \). Then \( \{V_{\alpha n}\} \) is a countable subcover for \( Y \). Thus \( Y \) is Lindelof space.

Theorem 4.13. Let \( f : X \rightarrow Y \) be a \( gagr \)- irresolute surjection and \( X \) be \( gagr \)-Lindelof, then \( Y \) is \( gagr \)-Lindelof Space.

Proof. Let \( f : X \rightarrow Y \) be a \( gagr \)- irresolute surjection and \( X \) be \( gagr \)-Lindelof. Let \( \{V_\alpha\} \) be an open cover for \( Y \). Then \( \{f^{-1}(V_\alpha)\} \) is a cover of \( X \) by \( gagr \)-open sets. Since \( X \) is \( gagr \)-Lindelof, \( \{f^{-1}(V_\alpha)\} \) contains a countable sub cover, namely \( \{f^{-1}(V_{\alpha n})\} \). Then \( \{V_{\alpha n}\} \) is a countable subcover for \( Y \). Thus \( Y \) is \( gagr \)-Lindelof space.

Theorem 4.14. If \( f : X \rightarrow Y \) is a \( gagr \)-open function and \( Y \) is \( gagr \)-Lindelof space, then \( X \) is Lindelof space.

Proof. Let \( \{V_\alpha\} \) be an open cover for \( X \). Then \( \{f(V_\alpha)\} \) is a cover of \( Y \) by \( gagr \)-open sets. Since \( Y \) is \( gagr \) Lindelof, \( \{f(V_\alpha)\} \) contains a countable sub cover, namely \( \{f(V_{\alpha n})\} \). Then \( \{V_{\alpha n}\} \) is a countable sub cover for \( X \). Thus \( X \) is Lindelof space.
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