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Numerical Solution of a 2-D Linearised Boussinesq Equation

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Abstract

The numerical solution of a 2-D linearized Boussinesq equation is presented and predict the spatio-temporal variation of the water table in a finite aquifer system in response to a transient recharge from rectangular basin. The boundary condition of the finite aquifer are taken as fixed heads, which would apply when the aquifer system is surrounded by open water bodies. Characteristic behaviours of the solution are illustrated with the help of a numerical example and solution is obtained by using finite difference method.

Mathematics Subject Classification: 93A30

Keywords: linearized Boussinesq equation, transient recharge, water table, numerical solution, finite difference method.

Notations:
A = length of aquifer in the x direction
B = length of aquifer in the y direction
$H = h^2 - h_0^2$
h= variable water table height
h_0= initial water table height
$a = \frac{K\bar{h}}{S_y}$
K= hydraulic conductivity

$\bar{h}$= weighted mean of the depth of saturation
S_y= specific yield
P= constant rate of recharge
$P_1 + P_0$= initial rate of transient recharge
P_1= final rate of transient recharge
t= time of observation since the beginning of recharge
x, y= space coordinates
$x_2 - x_1$= length of recharge basin
$y_2 - y_1$= width of recharge basin
β= decay constant

1 Introduction

Several types of models have been used to study groundwater flow system. These can be divided into three broad categories analog model, mathematical models and analytical and numerical models. Hydrological studies usually involve mathematical modeling of groundwater flow. Such model consist of a set of differential equations which govern the flow of groundwater. They have been in the use since the late 1800's.

To develop a numerical model of a physical system, it is first necessary to understand how that system behaves. This understanding takes the form of Darcy's law and concept of storage. These concepts and laws are then translated into mathematical expressions, usually partial differential equation with initial and boundary conditions. Numerical methods provide a means for solving these equations in their most general form.

Numerical solution normally involves approximating continuous partial differential equation with a set of discrete equations in time period of interest are divided in some fashion, resulting in an equation or set of equations for each subregion and time step. These discrete equations are combined to form a system of algebraic equations that must be solved for each time step. Finite difference and finite element methods are the major numerical techniques used in groundwater applications.

The finite difference method one of the oldest, most general applicable and most easily understood methods of obtaining numerical solution to steady and unsteady groundwater problems. Also, the finite difference method is a numerical method that can provide approximate solutions under general circumstances, when we solving partial differential groundwater equations.

Many 2-D analytical solutions of the Boussinesq equation have been used to predict water table fluctuations owing to uniform and constant recharge from basins of different configuration. In 1967, Hantush found analytical solutions for an infinite unconfined aquifer subject to recharge from rectangular and circular basins. Later on Rao and Sarma (1981 a, b) developed analytical solutions to describe water table fluctuations in finite aquifers to recharge from a rectangular area. In 1991, Zomorodi showed that solutions based on the assumption of constant recharge do not predict the decay of the water table owing to a declining recharge rate in field problem. In theoretical studies on water table fluctuations in 1-D aquifer systems many workers such as Rai and Singh (1981, 1992), Singh et al. (1991) have shown that the variation in recharge rate significantly affects the growth of the water table.

Singh and Rai (1980) have developed an approximate analytical solution for the fluctuations of the water table in a finite aquifer in response to an exponentially decaying rate of recharge. Rai and Singh (1995) have presented an analytical solution of the 2-D Boussinesq equation with an exponentially decreasing recharge rate which is predict water table fluctuation beneath a rectangular basin.

Sontakke and Rokade (2014) have developed a numerical solution of a 1-D linearised Boussinesq equation which describe water table fluctuation in the presence of time varying recharge from recharge basins for an one canal.

In the present work, a numerical solution of the 2-D linearised Boussinesq equation with an exponentially decreasing recharge rate is developed to predict water table fluctuation beneath a rectangular basin. The solution is obtained by using a finite difference method.

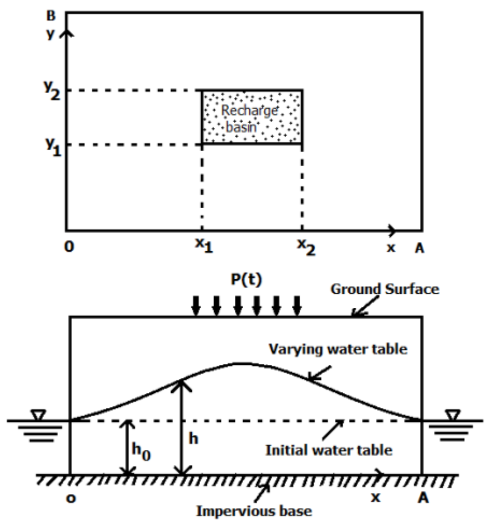

Figure 1. Schematic diagram of the recharge scheme and flow system.

2 Mathematical Formulation and Solution

A schematic diagram of the recharge scheme and a vertical cross section of the flow system are shown in above Figure 1. An unconfined aquifer with a horizontal impermeable base receives the recharge from an overlying rectangular basin. The aquifer is surrounded by open water bodies. It is assumed that, the water levels of the water bodies remain at a constant height equal to the initial depth of saturation, and the rate of recharge is so small compared with the hydraulic conductivity that water flows almost horizontally beneath the water table.

The groundwater movement in the flow system under consideration as shown in Figure 1 is represented by the following 2-D Boussinesq equation,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + 2\frac{P(t)}{K} = \frac{1}{a}\frac{\partial H}{\partial t} \tag{1}$$

The initial condition and boundary conditions are

$$H(x, y, 0) = 0 \tag{2}$$

$$H(0, y, t) = H(x, 0, t) = H(A, y, t) = H(x, B, t) = 0 \tag{3}$$

where $H = h^2 - h_0^2$, h is the variable water table height, h_o is the initial depth of saturation, $a = \frac{K\bar{h}}{S_y}$, K is the hydraulic conductivity, S_y is the specific yield, $\bar{h}$ is the weighted mean of the depth of saturation and can be evaluated by using the method of successive approximation, x and y are space coordinates, t is the time of observation, A and B are the aquifers dimensions in the x and y directions respectively, and P(t) is the time varying rate of recharge. Here the rate of recharge is considered as exponentially decreasing with time from an initial value $P_1 + P_0$ to a lower value P_1 as shown in Figure 2 and is represented as,

$$P(t) = \begin{cases} P_1 + P_0 e^{-\beta t} & , for\ x_1 \leq x \leq x_2,\ \ y_1 \leq y \leq y_2 \\ 0 & , elsewhere \end{cases} \tag{4}$$

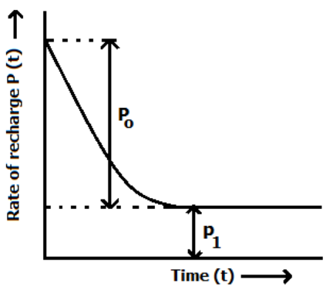

Figure 2. Time varying recharge rate.

The boundary value problem is solved by using the finite difference method. Thus using forward difference to the time derivative and central difference to the space derivative approximation, the linearised Boussinesq equation (1) yields,

$$\frac{H_{i,j}^{n+1} - H_{i,j}^n}{a\Delta t} = \frac{H_{i-1,j}^n - 2H_{i,j}^n + H_{i+1,j}^n}{(\Delta x)^2} + \frac{H_{i,j-1}^n - 2H_{i,j}^n + H_{i,j+1}^n}{(\Delta y)^2} + \frac{2P_{i,j}^n}{K}$$

Now, define $r = \frac{a\Delta t}{h^2}$, where $\Delta x = \Delta y = h$. Then the above equation reduces to

$$H_{i,j}^{n+1} = (1-4r)H_{i,j}^{n} + r[H_{i-1,j}^{n} + H_{i+1,j}^{n} + H_{i,j+1}^{n} + H_{i,j-1}^{n}] + \frac{2a\Delta t}{K}P_{i,j}^{n} \quad (5)$$

3 Numerical Results and Discussion

The numerical example of Rai and Singh (1995) is considered here to demonstrate the application of equation (5) which illustrate effect of variation in recharge rate on the water table fluctuation. In this example the controlling parameters are $A = B = 820$cm, $x_1 = y_1 = 390$cm, $x_2 = y_2 = 430$cm, $K = 0.4$cm/s, $S_y = 0.15$ and $h_0 = 10$cm. Numerical values of other new parameters are taken as $P_1 = 0.01$cm/s and $P_0 = 0.02$cm/s

Using equation (5), we computed water table variations at the centre of the recharge basin. Let us choose the value of $\Delta x = \Delta y = 205$ i.e h and $\Delta t = 30$, we get a numerically stable solution to satisfy the stability condition, $r = \frac{a\Delta t}{h^2} \leq \frac{1}{2}$, so that $r = 0.02$. Now, we shall find the mesh points. At the zeroth level($n = 0$) the initial condition at $t = 0$ and boundary conditions are $H_{i,j}^{0} = 0$, $i = j = 0, 1, 2, 3, 4$.

From given data, $\bar{h} = 10$cm, $K = 0.4$cm/s, $S_y = 0.15$, $\Delta t = 30$ then $a = \frac{K\bar{h}}{S_y} = 26.66$ and $\frac{2a\Delta t}{K} = 3999$. Also, using equation (4) we can find the values of $P_{i,j}^{n}(t)$ at different times in response to a constant rate of recharge $\beta = 0$ and exponentially decreasing recharge rate for $\beta = 0.01s^{-1}$ and $\beta = 0.02s^{-1}$.

Using above all values and equation (5) we calculate the water table variation at the centre of the recharge basin for $\beta = 0.02$ at the first level $n = 0$, $t = 30$ for $i, j = 1, 2, 3$, we get $H_{2,2}^{1} = 84$.

Similarly, we may obtain another values of the second and higher levels for $\beta = 0.02$.

Thus, we can use the simple algorithm described by equation (5) and compute further values of water table variations at the centre of the recharge basin for subsequent time steps at all nodes for $\beta = 0$ and $\beta = 0.01$, which are conveniently tabulated in following table.

n	t	β		
		0.02	0.01	0
0	0	0	0	0
0	30	84	100	120
1	60	141	176	230
2	90	182	234	332
3	120	215	280	426

From Figure 3 variation of the water table at the centre of recharge basin in response to a constant rate of recharge $\beta = 0$ is compared with the variation in response to the exponentially decreasing recharge rate for $\beta = 0.01s^{-1}$ and $\beta = 0.02s^{-1}$. From Fig. 3 it can be seen that for constant recharge rate the water table continuously rises. On the other hand, in the case of exponentially decreasing recharge rate, it rises in the beginning and, after attaining a maximum height, it starts declining. As expected the growth of the water table for smaller values of β is relatively greater than that for larger values of β.

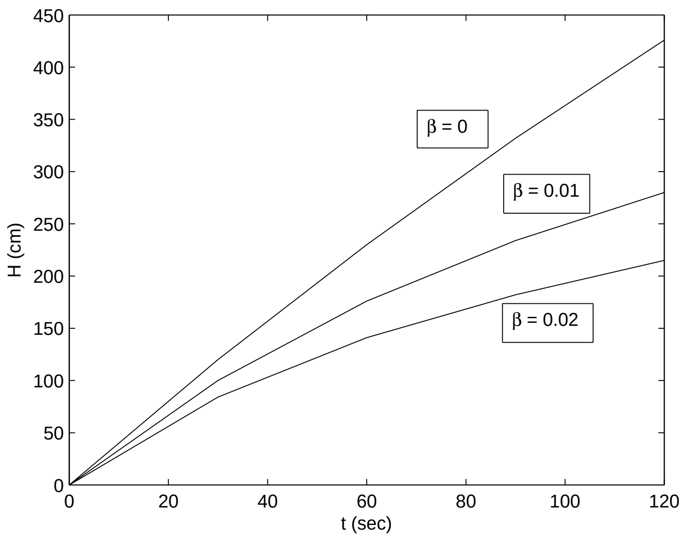

Figure 3. Water table variations at the centre of the recharge basin.

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