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Numerical Investigation of the Fuzzy Differential Inclusions using HHPM

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Abstract

In this article, the He's Homotopy Perturbation Method (HHPM) [8, 9, 10] is used to study the fuzzy differential inclusion [4, 7]. The obtained discrete solutions using the He's Homotopy Perturbation Method are compared with the exact solutions of the fuzzy differential inclusion and are found to be very accurate. Error graphs for discrete and exact solutions are presented in a graphical form to show the efficiency of this method.

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1 Introduction

The reachable set of a differential inclusion (the latter interpreted as a uncertain system) is the minimal guaranteed estimation of the current state. Therefore, to calculate reachable sets is a cornerstone of the estimation and control of uncertain systems [3]. A lot of work has been done for developing numerical approximation methods, see the surveys. Since the geometry of the reachable sets could be rather complicated, specific subclasses of sets are usually used as approximation tools: boxes, polyhedral sets, ellipsoids, box or polyhedral complexes, etc. In some cases convergence results are obtained, but usually to achieve a good approximation one has to use rather complex approximating sets. Even when analytical solutions can be found, they are not always useful in practice since the computational cost involved is very high. [1, 2, 5, 6]

Recently, E. Babolian, S. Abbasbandy and M. Alavi [4] presented the numerical solution of fuzzy differential inclusion by Euler method. S. Sekar and K. Prabhavathi [7] solved the same fuzzy differential inclusion using Leapfrog method. The objective of this article is to use the He's Homotopy Perturbation Method to solve the fuzzy differential inclusions (discussed by S. Sekar and K. Prabhavathi [7]).

2 He's Homotopy Perturbation Method

In this section, we briefly review the main points of the powerful method, known as the He's homotopy perturbation method [8, 9, 10]. To illustrate the basic ideas of this method, we consider the following differential equation:

$$A(u) - f(t) = 0, u(0) = u_0, t \in \Omega \tag{1}$$

where A is a general differential operator, u_0 is an initial approximation of Eq. (1), and $f(t)$ is a known analytical function on the domain of Ω. The operator A can be divided into two parts, which are L and N, where L is a linear operator, but N is nonlinear. Eq. (1) can be, therefore, rewritten as follows:

$$L(u) + N(u) - f(t) = 0$$

By the homotopy technique, we construct a homotopy $U(t,p) : \Omega \times [0,1] \to \Re$, which satisfies:

$$H(U,p) = (1-p)[LU(t) - Lu_0(t)] + p[AU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{2}$$

or

$$H(U,p) = LU(t) - Lu_0(t) + pLu_0(t) + p[NU(t) - f(t)] = 0, p \in [0,1], t \in \Omega \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, which satisfies the boundary conditions. Obviously, from Eqs. (2) or (3) we will have $H(U,0) = LU(t) - Lu_0(t) = 0, H(U,1) = AU(t) - f(t) = 0$.

The changing process of p from zero to unity is just that of $U(t,p)$ from $u_0(t)$ to $u(t)$. In topology, this is called homotopy. According to the He's Homotopy Perturbation method, we can first use the embedding parameter p

as a small parameter, and assume that the solution of Eqs. (2) or (3) can be written as a power series in p :

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + p^3 U_3 + ... \tag{4}$$

Setting $p = 1$, results in the approximate solution of Eq.(1)

$$U(t) = \lim_{p \to 1} U = U_0 + U_1 + U_2 + U_3 + ...$$

Applying the inverse operator $L^{-1} = \int_0^t (.)dt$ to both sides of Eq. (3), we obtain

$$U(t) = U(0) + \int_0^t Lu_0(t)dt - p\int_0^t Lu_0(t)dt - p[\int_0^t (NU(t) - f(t))dt] \tag{5}$$

where $U(0) = u_0$.

Now, suppose that the initial approximations to the solutions, $Lu_0(t)$, have the form

$$Lu_0(t) = \sum_{n=0}^{\infty} \alpha_n P_n(t) \tag{6}$$

where α_n are unknown coefficients, and $P_0(t), P_1(t), P_2(t), ...$ are specific functions. Substituting (4) and (6) into (5) and equating the coefficients of p with the same power leads to

$$\left.\begin{array}{r} p^0 : U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \\ p^1 : U_1(t) = -\sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt - \int_0^t (NU_0(t) - f(t))dt \\ p^2 : U_2(t) = -\int_0^t NU_1(t)dt \\ \vdots \\ p^j : U_j(t) = -\int_0^t NU_{j-1}(t)dt \end{array}\right\} \tag{7}$$

Now, if these equations are solved in such a way that $U_1(t) = 0$, then Eq. (7) results in $U_1(t) = U_2(t) = U_3(t) = \ldots = 0$ and therefore the exact solution can be obtained by using

$$U(t) = U_0(t) = u_0 + \sum_{n=0}^{\infty} \alpha_n \int_0^t P_n(t)dt \tag{8}$$

It is worth noting that, if $U(t)$ is analytic at $t = t_0$, then their Taylor series

$$U(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

can be used in Eq. (8), where $a_0, a_1, a_2, ...$ are known coefficients and α_n are unknown ones, which must be computed.

3 Fuzzy differential inclusions

Prior to introduce fuzzy differential inclusion we must denote fuzzy sets and fuzzy numbers as follows. We place a tilde over a symbol to denote a fuzzy set so $\tilde{X}, \tilde{A}, ...,$ all represent fuzzy subsets in R. We write $\tilde{X}(t)$ for the membership function of $\tilde{X}$ evaluated at $t \in R$. An α - cut of $\tilde{X}$ written $[\tilde{X}]_\alpha$, is defined as $\{t : \tilde{X}(t) \geq \alpha\}$, for $0 < \alpha < 1$ and $[\tilde{X}]_0 = \bigcup_{\alpha \in (0,1]} [\tilde{X}]_\alpha$. A triangular fuzzy number $\tilde{N}$ is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $\tilde{N}(t)$ is triangle with base on the interval $[a_1, a_3]$ and vertex at $t = a_2$ where $\tilde{N}(a_1) = \tilde{N}(a_3) = 0, \tilde{N}(a_2) = 1$, and We write $\tilde{N} = (a_1/a_2/a_3)$, For $x \in R^n$ and $A, B \subset r^n$ let

$$\rho(x, A) = inf\{|x - a|, a \in A\}$$

$$\beta(A, B) = sup\{\rho(a, B), a, a \in A\}$$

$$d_H(A, B) = max\{\beta(A, B), \beta(B, A)\}$$

The Hausdorff distance d_H defines a metric on the non empty and compact subsets of R^n. For two fuzzy sets $\tilde{A}, \tilde{B}$ the Hausdorff metric is defined as $\tilde{d}_H(\tilde{A}, \tilde{B}) = sup_{\alpha \in [0,1]} d_H([\tilde{A}]_\alpha, [\tilde{B}]_\alpha)$.

We can replace functions and initial values in the problem

$$\left.\begin{aligned} x'(t) = f(t, x(t)) \\ x(0) = x_0 \end{aligned}\right\} \tag{9}$$

by (7.4) set-valued functions which leads to the following differential inclusion (DI),

$$\left.\begin{aligned} x'(t) = F(t, x(t)) \\ x(0) = X_0 \end{aligned}\right\} \tag{10}$$

Where $F : [0, T] \times R^n \to 2^{R^n}/\{\phi\}$ is a set-valued function and $X_0 \subset R^n$ is compact and convex. A function $x : [0, T] \to R^n$ is a solution of (7.5) if it is an absolutely continuous and satisfies (7.5) almost everywhere. Let χ denote the set of all solutions of (7.5), the reachable set $X(t)$ at time $t \in [0, T]$ is defined as,

$$X(t) = \{x(t)/x \in \chi\}$$

The set $X(t)$ is the set of all possible solutions of (7.4) at time t. A reasonable generalization of this approach which takes vagueness into account is to replace sets by fuzzy sets, i.e. (7.5) becomes the fuzzy differential inclusion,

$$\left.\begin{aligned} x'(t) = \tilde{F}(t, x(t)) \\ x(0) = \tilde{X}_0 \end{aligned}\right\} \tag{11}$$

On $[0, T]$ with a fuzzy function $\tilde{F} : [0, T] \times R^n \to E^n$, where fuzzy set $\tilde{X}_0 \in E^n$ and E^n is the set of normal, upper semi-continuous, fuzzy convex and compactly supported fuzzy sets on R^n. Also $x'(t)$ is the usual crisp derivative of the crisp differentiable function $x(t)$ with respect to t. In this section we introduce a Leapfrog method for finding reachable set $\tilde{X}(t)$ that are based on the theoretical consideration of the following theorem.

Theorem 3.1. *Suppose the fuzzy function $\tilde{F} : [0,T] \times R^n \to E^n$ to be continuous in t and also satisfies Lipschitz condition $\tilde{d}_H(\tilde{F}(t,x), \tilde{F}(t,y)) \leq L|x-y|$ on R^n with Lipschitz $L > 0$. Consider the set $\tilde{\chi}$ of solutions to (7.6). The reachable set $\tilde{X}(t)$ associated with $\tilde{\chi}$ is a normal, upper semi-continuous and compactly supported fuzzy set for all $t \in [0,T]$. If $\tilde{F}$ is also concave, i.e., $\alpha\tilde{F}(t,x)+\beta\tilde{F}(t,y) \subset \tilde{F}(t,\alpha x+\beta y)$, For all $\alpha, \beta > 0$,$\alpha+\beta = 1$ then $\tilde{X}(t) \in E^n$.*

Now, consider the initial value problem (7.6) with $n = 1$, i.e.

$$\left.\begin{aligned} x'(t) = \tilde{F}(t,x(t)) \\ x(0) = \tilde{X}_0 \end{aligned}\right\} \tag{12}$$

On $J = [0,T]$ with a fuzzy concave function $\tilde{F} : J \times R \to E$, where fuzzy set $\tilde{X} \in E$ and the hypotheses of Theorem 7.9.1 are satisfied. We call a function $x_\alpha : J \to R$ an α - solution to (7.7), if it is absolutely continuous and satisfies

$$\left.\begin{aligned} x'_\alpha(t) = \tilde{F}_\alpha(t,x(t)) \\ x_\alpha(0) = [\tilde{X}_0]_\alpha \end{aligned}\right\} \tag{13}$$

Almost everywhere on J, where $F_\alpha(t,x(t))$ is the α - cut of the fuzzy set $\tilde{F}(t,x(t))$. The set of all α - solution to (7.8) is denoted by χ_α, and the α -reachable set $X_\alpha(t)$ is defined as $X_\alpha(t) = \{x(t) : x \in \chi_\alpha\}$. In this section, the α -reachable set $X_\alpha(t)$ is approximated by Leapfrog method.

4 Numerical Example for the fuzzy differential inclusions

In this section, we apply He's Homotopy Perturbation Method to solve one dimensional fuzzy differential inclusion. The main objective here is to solve this example using the He's Homotopy Perturbation Method given in Section 2 and compare our results with the presented results in S. Sekar and K. Prabhavathi [7].

The discrete solutions obtained by the two methods, He's Homotopy Perturbation Method and Leapfrog method. The absolute errors between them are tabulated and are presented in Figures 1 – 2. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of $'r'$ and are presented in Figure 1 to Figure 2 for the following problem, using three dimensional effects.

Figure 1: Error estimation of 0 -reachable set $[\tilde{X}_0]_0$ of Example 4.1

4.1 Example

Consider the fuzzy differential inclusions on R^+,

$$\left.\begin{array}{r} x'(t) \in -x(t) + \tilde{e}cost \\ x(0) \in \tilde{X}_0 \end{array}\right\} \tag{14}$$

where $\tilde{e}$ and $\tilde{X}_0$ are symmetric triangular fuzzy numbers with level sets $[\tilde{e}]_\alpha = [0.05(\alpha - 1), 0.05(1-\alpha)]$ and $[\tilde{X}_0]_\alpha = [0.05(\alpha-1), 0.05(1-\alpha)]$. The α–solution set is given for $t \geq 0$ by

$$x_\alpha(t) \in \frac{1}{2}(sint + cost)[\tilde{e}]_\alpha + [[\tilde{X}_0]_\alpha - \frac{1}{2}[\tilde{e}]_\alpha]e^{-t}$$

Now, we obtain the approximation using single-term Haar wavelet series and Leapfrog method of 0-reachable set and calculated error in Figure 1 and $\tilde{X}(5)$ in Figure 2 with $\triangle t = 0.01$.

From the graphical representation is given for the one dimensional fuzzy differential equation based on fuzzy differential inclusion shows that single-term Haar wavelet series solutions [S. Sekar and K. Prabhavathi [7]] have little error in the all the stages but Leapfrog method approximate solutions match well in all stages.

5 Conclusion

The obtained approximate solutions for the one dimensional fuzzy differential equation based on fuzzy differential inclusion using the He's Homotopy Per-

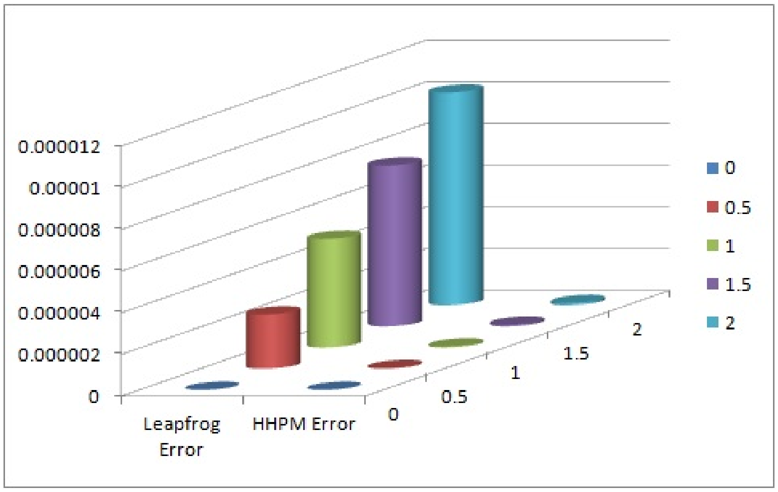

Figure 2: Error estimation of $\tilde{X}(5)$ of Example 4.1

turbation Method give more accurate values when compared to the Leapfrog method discussed by [S. Sekar and K. Prabhavathi [7]]. From the error graph presented in Figures 1 - 2, an elaborate, well composed comparison has been carried out with the aid of the obtained results and graphs. One can observe that the He's Homotopy Perturbation Method yields very less error when compared to Leapfrog method and hence this He's Homotopy Perturbation Method is more suitable for studying the one dimensional fuzzy differential equation based on fuzzy differential inclusion.

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