Abstract

The identities on idempotent derivations of $n$-Lie algebras are provided, and the structure of $n$-Lie algebras which have idempotent derivations is discussed. The method of constructing $n$-Lie algebras which with idempotent derivations by $n$-Lie algebras and modules over the complex field is introduced.

Mathematics Subject Classification: 17B05, 17B30.

Keywords: $n$-Lie algebra, derivation, idempotent derivation

1 Fundamental notion

In this paper we investigate the structure of $n$-Lie algebras [1] with idempotent derivations over the complex field for $n \geq 3$. First we introduce some basic notions [2, 3].

An $n$-Lie algebra is a vector space $A$ endowed with an $n$-ary multi-linear skew-symmetric operation $[\cdot, \ldots, \cdot]$ satisfying the $n$-Jacobi identity, that is, for all $x_1, \ldots, x_n, y_2, \ldots, y_n \in A$,

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n].$$  \hfill (1)
Let $I$ be a subspace of $n$-Lie algebra $A$, if $I$ satisfies $[I,\cdots,I] \subseteq I$ ($[I,A,\cdots,A] \subseteq I$) then $I$ is called a subalgebra (an ideal) of $A$. If an ideal $I$ satisfies $[I,I,A,\cdots,A] = 0$, then $I$ is called an abelian ideal.

A derivation of $A$ is a linear map $D$ of $A$ satisfying for all $x_1,\cdots,x_n \in A$,

$$D([x_1,\cdots,x_n]) = \sum_{i=1}^{n} [x_1,\cdots,D(x_i),\cdots,x_n].$$

(2)

If a derivation $D$ satisfies $D^2 = D$, then $D$ is called an idempotent derivation of $A$.

## 2 Main results

In the following, we suppose $A$ is a finite dimensional $n$-Lie algebra over the complex field $F$ ($n \geq 3$). We first prove some identities on idempotent derivations.

**Lemma 1** Let $A$ be an $n$-Lie algebra, and $D$ be an idempotent derivation, then for all $x_1,\cdots,x_n \in A$, we have

1) $\sum_{1 \leq i < j \leq n} [x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n] = 0$.

2) $D([x_1,\cdots,D(x_i),\cdots,x_n]) = [x_1,\cdots,D(x_i),\cdots,x_n]$.

3) $\sum_{i=1}^{n} [D(x_1),\cdots,D(x_{i-1}),x_i,D(x_{i+1}),\cdots,D(x_n)] = 0$.

4) $[D(x_1),\cdots,D(x_n)] = 0$.

5) $[x_1,\cdots,x_i,D(x_{i+1}),\cdots,D(x_n)] = 0, 1 \leq i \leq n - 2$.

**Proof** By identity (2) and $D = D^2$, for all $x_1,\cdots,x_n \in A$,

$$D([x_1,\cdots,x_n]) = D^2([x_1,\cdots,x_n])$$

$$= \sum_{i=1}^{n} [x_1,\cdots,D^2(x_i),\cdots,x_n] + \sum_{1 \leq i < j \leq n} [x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n]$$

$$= D([x_1,\cdots,x_n]) + \sum_{1 \leq i < j \leq n} [x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n].$$

It follows the result 1).

Thanks to the result 1) and

$$D([x_1,\cdots,D(x_i),\cdots,x_n])$$

$$= [x_1,\cdots,D^2(x_i),\cdots,x_n] + \sum_{1 \leq i < j \leq n} [x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n]$$

$$= [x_1,\cdots,D^2(x_i),\cdots,x_n]$$

we obtain the result 2).

By the result 1), we have

$$\sum_{1 \leq i < j \leq n} D([x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n])$$

$$= 2 \sum_{1 \leq i < j \leq n} [x_1,\cdots,D(x_i),\cdots,D(x_j),\cdots,x_n]$$
The result 3) follows from the induction.

Similarly, we have

\[
\sum_{1 \leq i < j < k \leq n} [x_1, \ldots, D(x_i), \ldots, D(x_j), \ldots, D(x_k), \ldots, x_n] = 0.
\]

The result 3) follows from the induction.

By the identity (2) and the above discussion,

\[
[D(x_1), \ldots, D(x_n)] = [x_1, D(x_2), \ldots, D(x_n)] - (n - 1)[x_1, D(x_2), \ldots, D(x_n)]
\]

\[
= D([D(x_1), x_2, D(x_3), \ldots, D(x_n)]) - (n - 1)[D(x_1), x_2, D(x_3), \ldots, D(x_n)]
\]

\[
= D([D(x_1), \ldots, D(x_{i-1}), x_i, D(x_{i+1}), \ldots, D(x_n)])
\]

\[
- (n - 1)[D(x_1), \ldots, D(x_{i-1}), x_i, D(x_{i+1}), \ldots, D(x_n)]
\]

\[
= D([D(x_1), \ldots, D(x_{n-1}), x_n]) - (n - 1)[D(x_1), \ldots, D(x_{n-1}), x_n].
\]

Again by the result 3), we have

\[
n[D(x_1), \ldots, D(x_n)]
\]

\[
= D(\sum_{i=1}^n [D(x_1), \ldots, D(x_{i-1}), x_i, D(x_{i+1}), \ldots, D(x_n)])
\]

\[
- (n - 1) \sum_{i=1}^n [D(x_1), \ldots, D(x_{i-1}), x_i, D(x_{i+1}), \ldots, D(x_n)] = 0.
\]

The result 4) holds.

Thanks to the result 4) and result 2),

\[
D([x_1, D(x_2), \ldots, D(x_n)]) = [D(x_1), \ldots, D(x_n)] + (n - 1)[x_1, D(x_2), \ldots, D(x_n)]
\]

\[
= (n - 1)[x_1, D(x_2), \ldots, D(x_n)].
\]

Thanks to \(chF = 0\), \([x_1, D(x_2), \ldots, D(x_n)] = 0\).

Now suppose \([x_1, \ldots, x_{i-1}, D(x_i), \ldots, D(x_n)] = 0\) holds for \(i\), where \(1 \leq i \leq n - 1\). Then

\[
D([x_1, \ldots, x_{i-1}, x_i, D(x_{i+1}), \ldots, D(x_n)])
\]

\[
= \sum_{j=1}^i [x_1, \ldots, D(x_j), \ldots, x_i, D(x_{i+1}), \ldots, D(x_n)]
\]

\[
+ (n - i)[x_1, \ldots, x_{i-1}, x_i, D(x_{i+1}), \ldots, D(x_n)]
\]

\[
= (n - i)[x_1, \ldots, x_{i-1}, x_i, D(x_{i+1}), \ldots, D(x_n)].
\]

We obtain \([x_1, \ldots, x_{i-1}, x_i, D(x_{i+1}), \ldots, D(x_n)] = 0\) for all \(1 \leq i \leq n - 2\). The proof is completed.

**Lemma 2** Let \(A\) be an \(n\)-Lie algebra, and \(D\) be an idempotent derivation, then there exists a basis \(\{v_1, \ldots, v_r, u_1, \ldots, u_s\}\) of \(A\) such that

\[
\left\{\begin{array}{c}
D(v_i) = v_i, 1 \leq i \leq r, \\
D(u_j) = 0, 1 \leq j \leq s;
\end{array}\right.
\]

and the image \(D(A) = I = \sum_{i=1}^r Fv_i\) is an ideal, and the kernel \(K = \text{Ker} D =\)
Theorem 1 Let $A$ be an $n$-Lie algebra. Then there exists an idempotent derivation on $A$ if and only if $A = I \oplus K$, where $I$ is an abelian ideal and $K$ is a subalgebra.

Proof If there exists an idempotent derivation $D$ on the $n$-Lie algebra $A$, thanks to Lemma 2, $A = I \oplus K$, where $I = D(A)$ is an ideal and $K = \text{Ker} D$ is a subalgebra. For all $x_1, \ldots, x_r \in I$, $y_1, \ldots, y_{n-r} \in K$, where $r \geq 2$, since $[x_1, \ldots, x_r, y_1, \ldots, y_{n-r}] \in I$, by identities (2) and (3) and Lemma 2,

$$
[x_1, \ldots, x_r, y_1, \ldots, y_{n-r}] = \sum_{i=1}^{r} D([x_1, \ldots, x_r, y_1, \ldots, y_{n-r}])
$$

$$
+ \sum_{j=1}^{n-r} [x_1, \ldots, x_r, y_1, \ldots, D(y_j), \ldots, y_{n-r}]
$$

$$
= r [x_1, \ldots, x_r, y_1, \ldots, y_{n-r}] .
$$

Since $r \geq 2$ and $chF = 0$, we have $[x_1, \ldots, x_r, y_1, \ldots, y_{n-r}] = 0$. Therefore, $I = D(A)$ is an abelian ideal.

Conversely, define $D : A \to A$ as follows,

$$
D(x) = x, \quad D(y) = 0, \text{ for all } x \in I, \ y \in K .
$$

Obviously, $D$ satisfies $D^2 = D$, and for all $x_1, \ldots, x_n \in I, y_1, \ldots, y_n \in K$, and $2 \leq r \leq n-1$,

$$
D([y_1, \ldots, y_n]) = \sum_{i=1}^{n} [y_1, \ldots, D(y_i), \ldots, y_n] = 0 ,
$$

$$
D([x_1, \ldots, x_n]) = [x_1, \ldots, x_n] = n[x_1, \ldots, x_n] = \sum_{i=1}^{n} [x_1, \ldots, D(x_i), \ldots, x_n] = 0 ,
$$

$$
D([x_1, \ldots, x_r, y_1, \ldots, y_{n-r}]) = [x_1, \ldots, x_r, y_1, \ldots, y_{n-r}] = 0 ,
$$

$$
\sum_{i=1}^{r} [x_1, \ldots, D(x_i), \ldots, x_r, y_1, \ldots, y_{n-r}] = 0 .
$$

\[\sum_{j=1}^{n-r} [x_1, \ldots, x_r, y_1, \ldots, D(y_j), \ldots, y_{n-r}].\]
\[ + \sum_{j=1}^{n-r} [x_1, \cdots, x_r, y_1, \cdots, D(y_j), \cdots, y_{n-r}] \]
\[ = r [x_1, \cdots, x_r, y_1, \cdots, y_{n-r}] = 0. \]

Therefore, \( D \) is an idempotent derivation of \( A \). The proof is completed.

From Theorem 1, for all \( n \)-Lie algebra \( (A, [\cdots, \cdot, \cdots]) \) and an \( A \)-module \( (M, \rho) \). We can construct an \( n \)-Lie algebra \( B \) such that \( B \) with an idempotent derivation. Set \( B = A \oplus M \). By paper [4], \( B \) is an \( n \)-Lie algebra such that \( (A, [\cdots, \cdot, \cdots]) \) is a subalgebra of \( B \) and \( M \) is an abelian ideal, and for all \( x_1, \cdots, x_{n-1} \in A, m \in M, [x_1, \cdots, x_{n-1}, m]_B = \rho(x_1, \cdots, x_{n-1})(m) \). Then \( D : B \to B \) defined by \( D(x) = 0, D(m) = m \) for all \( x \in A \) and \( m \in M \), is an idempotent derivation of \( B \).

**Acknowledgements**

The first author (R.-P. Bai) was supported in part by the Natural Science Foundation (11371245) and the Natural Science Foundation of Hebei Province (A2014201006).

**References**


Received: January, 2015