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New characterizations of regular ordered semigroups in terms of fuzzy soft ideals

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Abstract

In this paper, some new characterizations of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals are obtained.

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1 Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as the probability theory, the theory of (intuitionistic) fuzzy sets, the theory of vague sets, the theory of interval mathematics, and the theory of rough sets. However, all of these have their advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterization tools. To overcome these difficulties, Molodtsov [11] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. This theory has proven useful in many different fields such as decision making, data analysis, forecasting and so on.

Up to the present, research on soft sets has been very active and many important results have been achieved in the theoretical aspect. Maji et al. [10] introduced several algebraic operations in soft set theory and published a detail theoretical study on soft sets. Ali et al. [3] further presented and investigated some new algebraic operations for soft sets. Aygünoğlu and Aygün [1] discussed the applications of fuzzy soft sets to group theory and investigated (normal) fuzzy soft groups. Feng et al. [2] investigated soft semirings by using the soft set theory. Jun [4] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [6] and Jun et al. [5] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras, respectively. Koyuncu and Tanay [8] introduced and studied soft rings. Zhan and Jun [15] characterized the (implicative, positive implicative and fantastic) filteristic soft BL-algebras based on $\in$-soft sets and q-soft sets. Yin et al. [14] introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals and investigated some of their related properties. They also derive some characterizations of left quasi-regular ordered semigroups and ordered semigroups that are left quasi-regular and intra-regular in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals.

As a continuation of the work of Yin et al. [14], this paper investigates some new characterizations of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper. The characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals is discussed in Section 3. Some conclusions are given in the last Section.

2 Preliminaries

2.1 Ordered semigroups

In this section, we recall some basic notions and results on ordered semigroups (see [7]).

An *ordered semigroup* is an algebraic system $(S, \cdot, \leq)$ consisting of a non-empty set S together with a binary operation " $+$ " and a compatible ordering " $\leq$ " on S such that $(S, \cdot)$ is a semigroup and $x \leq y$ implies $ax \leq ay$ and $xa \leq ya$ for all $x, y, a \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A subset I of S is called a *left* (resp., *right*) *ideal* of S if it satisfies the following conditions: (1) $SI \subseteq I$ (resp., $IS \subseteq I$); (2) if $x \in I$ and $S \ni y \leq x$, then $y \in I$. If I is both a left and a right ideal of S, then I is called an *ideal* of S.

A subset P of S is called a *bi-ideal* if it satisfies the following conditions: (1) $PP \subseteq P$; (2) $PSP \subseteq P$; (3) if $x \in P$ and $S \ni y \leq x$, then $y \in P$.

For $X, Y \subseteq S$, denote $(X] := \{x \in S | x \leq y \text{ for some } y \in X\}$ and $XY := \{xy \in S | x \in X, y \in Y\}$. For $x \in S$, define $A_x = \{(y, z) \in S \times S | x \leq yz\}$.

For $X, Y \subseteq S$, we have $X \subseteq (X]$, $(X](Y] \subseteq (XY]$, $((X]] = (X]$ and $(X] \subseteq (Y]$ if $X \subseteq Y$. X is said to be *idempotent* if $(X] = (X^2]$.

2.2 Fuzzy sets

Let X be a non-empty set. A fuzzy subset μ of X is defined as a mapping from X into $[0, 1]$, where $[0, 1]$ is the usual interval of real numbers. We denote by $\mathscr{F}(X)$ the set of all fuzzy subsets of X.

A fuzzy subset μ of X of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a *fuzzy point with support x and value r* and is denoted by x_r, where $r \in (0, 1]$.

In what follows let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For any $Y \subseteq X$, we define $\chi^{\delta}_{\gamma Y}$ be the fuzzy subset of X by $\chi^{\delta}_{\gamma Y}(x) \geq \delta$ for all $x \in Y$ and $\chi^{\delta}_{\gamma Y}(x) \leq \gamma$ otherwise. Clearly, $\chi^{\delta}_{\gamma Y}$ is the characteristic function of Y if $\gamma = 0$ and $\delta = 1$.

For a fuzzy point x_r and a fuzzy subset μ of X, we say that

(1) $x_r \in_{\gamma} \mu$ if $\mu(x) \geq r > \gamma$.
(2) $x_r \, q_{\delta} \, \mu$ if $\mu(x) + r > 2\delta$.
(3) $x_r \in_{\gamma} \vee q_{\delta} \mu$ if $x_r \in_{\gamma} \mu$ or $x_r \, q_{\delta} \, \mu$.

Let us now introduce a new ordering relation on $\mathscr{F}(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $\mu, \nu \in \mathscr{F}(X)$, by $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta\ \nu$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, μ and ν are said to be (γ,δ)-*equal*, denoted by $\mu =_{(\gamma,\delta)} \nu$, if $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \mu$.

In the sequel, unless otherwise stated, $\overline{\alpha}$ means α does not hold, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \subseteq \vee q_{(\gamma,\delta)}\}$.

Lemma 2.1 [14] *Let $\mu, \nu \in \mathscr{F}(X)$. Then $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ if and only if $\max\{\nu(x), \gamma\} \geq \min\{\mu(x), \delta\}$ for all $x \in X$.*

Lemma 2.2 [14] *Let $\mu, \nu, \omega \in \mathscr{F}(X)$. If $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \omega$, then $\mu \subseteq \vee q_{(\gamma,\delta)} \omega$.*

Lemmas 2.1 and 2.2 give that "$=_{(\gamma,\delta)}$" is an equivalence relation on $\mathscr{F}(X)$. It is also worth noticing that $\mu =_{(\gamma,\delta)} \nu$ if and only if $\max\{\min\{\mu(x), \delta\}, \gamma\} = \max\{\min\{\nu(x), \delta\}, \gamma\}$ for all $x \in X$ by Lemma 2.1.

Definition 2.3 [7] Let $(S, \cdot, \leq)$ be an ordered semigroup and $\mu, \nu \in \mathscr{F}(S)$. Define the *product* of μ and ν, denoted by $\mu \circ \nu$, by

$$(\mu \circ \nu)(x) = \begin{cases} \sup\limits_{(y,z) \in A_x} \min\{\mu(y), \nu(z)\} & \text{if there exist } y, z \in S \text{ such that } (y, z) \in A_x, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$.

Lemma 2.4 [14] *Let $(S, \cdot, \leq)$ be an ordered semigroup and $X, Y \subseteq S$. Then we have*

(1) $X \subseteq Y$ if and only if $\chi^\delta_{\gamma X} \subseteq \vee q_{(\gamma,\delta)} \chi^\delta_{\gamma Y}$.

(2) $\chi^\delta_{\gamma X} \cap \chi^\delta_{\gamma Y} =_{(\gamma,\delta)} \chi^\delta_{\gamma (X \cap Y)}$.

(3) $\chi^\delta_{\gamma X} \circ \chi^\delta_{\gamma Y} =_{(\gamma,\delta)} \chi^\delta_{\gamma (XY]}$.

2.3 Fuzzy soft sets

Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. As a generalization of soft set introduced in Molodtsov [11], Maji et al. [9] defined fuzzy soft set in the following way.

Definition 2.5 A pair $\langle F, A \rangle$ is called a *fuzzy soft set* over U, where $A \subseteq E$ and F is a mapping given by $F : A \to \mathscr{F}(U)$.

In general, for every $\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called *fuzzy value set* of parameter ε. The set of all fuzzy soft sets over U with parameters from E is called a *fuzzy soft class*, and it is denote by $\mathscr{F}\mathscr{S}(U, E)$.

Definition 2.6 [3] The *extended intersection* of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is a fuzzy soft set denoted by $\langle H, C\rangle$, where $C = A \cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C\rangle = \langle F, A\rangle \tilde{\cap} \langle G, B\rangle$.

Definition 2.7 [14] Let $V \subseteq U$. A fuzzy soft set $\langle F, A\rangle$ over V is said to be *a relative whole* (γ, δ)*-fuzzy soft set* (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if $F(\varepsilon) = \chi^{\delta}_{\gamma V}$ for all $\varepsilon \in A$.

Definition 2.8 [14] Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two fuzzy soft sets over U. We say that $\langle F, A\rangle$ is an (γ, δ)*-fuzzy soft subset* of $\langle G, B\rangle$ and write $\langle F, A\rangle \Subset_{(\gamma,\delta)} \langle G, B\rangle$ if

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.

$\langle F, A\rangle$ and $\langle G, B\rangle$ are said to be (γ, δ)*-fuzzy soft equal* and write $\langle F, A\rangle \asymp_{(\gamma,\delta)} \langle G, B\rangle$ if $\langle F, A\rangle \Subset_{(\gamma,\delta)} \langle G, B\rangle$ and $\langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$.

Lemma 2.9 [14] *Let $\langle F, A\rangle$, $\langle G, B\rangle$ and $\langle H, C\rangle$ be fuzzy soft sets over U. If $\langle F, A\rangle \Subset_{(\gamma,\delta)} \langle G, B\rangle$ and $\langle G, B\rangle \Subset_{(\gamma,\delta)} \langle H, C\rangle$. Then $\langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, C\rangle$.*

Definition 2.10 [14] The *product* of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over an ordered semigroup $(S, \cdot, \leq)$ is a fuzzy soft set over S, denoted by $\langle F \circ G, C\rangle$, where $C = A \cup B$ and

$$(F \circ G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \circ G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F \circ G, C\rangle = \langle F, A\rangle \odot \langle G, B\rangle$.

The following results can be easily deduced.

Lemma 2.11 [14] *Let $\langle F_1, A\rangle$, $\langle F_2, A\rangle$, $\langle G_1, B\rangle$ and $\langle G_2, B\rangle$ be fuzzy soft sets over an ordered semigroup $(S, \cdot, \leq)$ such that $\langle F_1, A\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle$ and $\langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle G_2, B\rangle$. Then*

(1) $\langle F_1, A\rangle \odot \langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle \odot \langle G_2, B\rangle$.

(2) $\langle F_1, A\rangle \tilde{\cap} \langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle \tilde{\cap} \langle G_2, B\rangle$.

3 Some new characterizations of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals

In this section, we will investigate the new characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals. Let us begin with formulating some results about $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideals over ordered semigroups.

Definition 3.1 [14] A fuzzy soft set $\langle F, A\rangle$ over an ordered semigroup $(S, \cdot, \leq)$ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-*fuzzy soft left* (resp., *right*) *ideal* over S if it satisfies:

(F1a) $\Sigma(S, A) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$ (resp., $\langle F, A\rangle \odot \Sigma(S, A) \Subset_{(\gamma,\delta)} \langle F, A\rangle$),
(F2a) If $y \leq x$, then $x_r \in_\gamma F(\varepsilon) \Rightarrow y_r \in_\gamma \vee q_\delta\ F(\varepsilon)$ for all $x, y \in S$, $\varepsilon \in A$ and $r \in (\gamma, 1]$.

A fuzzy soft set over S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-*fuzzy soft ideal* over S if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal over S.

Definition 3.2 [14] A fuzzy soft set $\langle F, A\rangle$ over an ordered semigroup $(S, \cdot, \leq)$ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-*fuzzy soft bi-ideal* over S if it satisfies conditions (F2a) and

(F3a) $\langle F, A\rangle \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$,
(F4a) $\langle F, A\rangle \odot \Sigma(S, A) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$.

Combing the results obtained in Yin et al. [14], we have the following results.

Lemma 3.3 *Let $\langle F, A\rangle$ be a fuzzy soft set over an ordered semigroup $(S, \cdot, \leq)$. Then $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (resp., right) ideal over S if and only if the following conditions hold:*

(F1b) $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$ *(resp.,* $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(x), \delta\}$*) for all $x, y \in S$ and $\varepsilon \in A$.*

(F2b) $y \leq x \Rightarrow \max\{F(\varepsilon)(y), \gamma\} \geq \min\{F(\varepsilon)(x), \delta\}$ *for all $x, y \in S$ and $\varepsilon \in A$.*

Lemma 3.4 *Let $\langle F, A\rangle$ be a fuzzy soft set over an ordered semigroup $(S, \cdot, \leq)$. Then $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal over S if and only if (F2b) and the following conditions hold:*

(F3a) $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ *for all $x, y \in S$ and $\varepsilon \in A$.*

(F4a) $\max\{F(\varepsilon)(xyz), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(z), \delta\}$ *for all $x, y, z \in S$ and $\varepsilon \in A$.*

Definition 3.5 [7] An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if for every $x \in S$ there exists $y \in S$ such that $x \leq xyx$. Equivalent definitions: (1) $x \in (xSx]$ $\forall x \in S$, (2) $A \subseteq (ASA]$ $\forall A \subseteq S$.

Lemma 3.6 [7,13] *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is regular if and only if one of the following conditions holds:*
(1) $L \cap R \subseteq (LR]$ for every left ideal L and every right ideal R of S;
(2) $B = (BSB]$ for every bi-ideal B of S.

Theorem 3.7 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is regular if and only if $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle G, B\rangle \odot \langle F, A\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle F, A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal $\langle G, B\rangle$ over S.*

Proof. Let S be regular, $\langle F, A\rangle$ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal and $\langle G, B\rangle$ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal over S, respectively. Now let x be any element of S, $\varepsilon \in A \cup B$ and $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle = \langle H, A \cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A - B$. Then $H(\varepsilon) = F(\varepsilon) = (G \circ F)(\varepsilon)$.

Case 2: $\varepsilon \in B - A$. Then $H(\varepsilon) = G(\varepsilon) = (G \circ F)(\varepsilon)$.

Case 3: $\varepsilon \in A \cap B$. Then $H(\varepsilon) = G(\varepsilon) \cap F(\varepsilon)$ and $(G \circ F)(\varepsilon) = G(\varepsilon) \cap F(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon) \circ F(\varepsilon)$. Since S is intra-regular, there exist $y \in S$ such that $x \leq xyx$. Then we have

$$\begin{aligned}
\max\{(G(\varepsilon) \circ F(\varepsilon))(x), \gamma\} &= \max\left\{ \sup_{(a,b)\in A_x} \min\{G(\varepsilon)(a), F(\varepsilon)(b)\}, \gamma \right\} \\
&\geq \max\{\min\{G(\varepsilon)(xy), F(\varepsilon)(x)\}, \gamma\} \\
&= \min\{\max\{G(\varepsilon)(xy), \gamma\}, \max\{F(\varepsilon)(x), \gamma\}\} \\
&\geq \min\{\min\{G(\varepsilon)(x), \delta\}, \min\{F(\varepsilon)(x), \delta\}\} \\
&= \min\{(F(\varepsilon) \cap G(\varepsilon)(x), \delta\}.
\end{aligned}$$

It follows that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon) \circ F(\varepsilon)$, that is, $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} (G \circ F)(\varepsilon)$.

Thus, in any case, we have $H(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} (G \circ F)(\varepsilon)$ and so $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle G, B\rangle \odot \langle F, A\rangle$.

Now, assume that the given condition holds, let L and R be any left ideal and any right ideal of S, respectively. It is easy to see that $\Sigma(L, E)$ and $\Sigma(R, E)$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal over S, respectively. Now, by the assumption, we have $\Sigma(L, E) \widetilde{\cap} \Sigma(R, E) \Subset_{(\gamma,\delta)} \Sigma(R, E) \odot_h \Sigma(L, E)$. Hence by Lemma 2.4, we have

$$\chi^{\delta}_{\gamma(L\cap R)} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma L} \cap \chi^{\delta}_{\gamma R} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma R} \odot \chi^{\delta}_{\gamma L} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma \overline{RL}}.$$

It follows from Lemma 2.4 that $L \cap R \subseteq \overline{RL}$. Therefore S is regular by Lemma 3.6. $\square$

Theorem 3.8 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is regular if and only if $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle F, A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle G, B\rangle$ over S.*

Proof. Assume that S is regular. Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S, respectively. Hence we have

$$\langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot \Sigma(S, B) \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$$

and

$$\langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \Sigma(S, A) \odot \langle G, B\rangle \odot \Sigma(S, A) \Subset_{(\gamma,\delta)} \langle G, B\rangle.$$

Thus $\langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \widetilde{\cap} \langle G, B\rangle$. Now, analogous to the proof of Theorem 3.7, to prove $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle$, it suffices to show that for any $x \in S$ and $\varepsilon \in A \cap B$, we have that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon) \circ G(\varepsilon) \circ F(\varepsilon)$. In fact, since S is regular, there exist $y \in S$ such that $x \leq xyx \leq yxzyxzx$, and then we have

$$\begin{aligned}
&\max\{(F(\varepsilon) \circ G(\varepsilon) \circ F(\varepsilon))(x), \gamma\}\\
&= \max\left\{\sup_{(a,b)\in A_x} \min\{(F(\varepsilon) \circ G(\varepsilon))(a), G(\varepsilon)(b)\}, \gamma\right\}\\
&\geq \max\{\min\{(F(\varepsilon) \circ G(\varepsilon))(xy), G(\varepsilon)(x)\}, \gamma\}\\
&= \max\left\{\min\left\{\sup_{(a,b)\in A_{xy}} \min\{F(\varepsilon)(a), G(\varepsilon)(b)\}, G(\varepsilon)(x)\right\}, \gamma\right\}\\
&\geq \max\left\{\min\left\{\min\{F(\varepsilon)(x), G(\varepsilon)(yxy)\}, G(\varepsilon)(x)\right\}, \gamma\right\}\\
&(\text{since } xy \leq xyxy)\\
&= \min\{\max\{F(\varepsilon)(x), \gamma\}, \max\{G(\varepsilon)(yxy), \gamma\}, \max\{G(\varepsilon)(x), \gamma\}\}\\
&\geq \min\{\min\{F(\varepsilon)(x), \delta\}, \min\{G(\varepsilon)(x), \delta\}\}\\
&= \min\{(F(\varepsilon) \cap G(\varepsilon)(x), \delta\},
\end{aligned}$$

as required. Therefore, $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle$.

Now, assume that $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle F, A\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle F, A\rangle$ and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle G, B\rangle$ over S. Let B be any bi-ideal of S. It is easy to see that $\Sigma(B, E)$ and $\Sigma(S, E)$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal over S, respectively. Now, by the assumption and Lemma 2.4, we have

$$\chi^\delta_{\gamma(B\cap S)} =_{(\gamma,\delta)} \chi^\delta_{\gamma B} \cap \chi^\delta_{\gamma S} =_{(\gamma,\delta)} \chi^\delta_{\gamma B} \odot \chi^\delta_{\gamma S} \odot \chi^\delta_{\gamma B} =_{(\gamma,\delta)} \chi^\delta_{\gamma \overline{BSB}}.$$

It follows from Lemma 2.4 that $B = \overline{BSB}$. Therefore S is regular by Lemma 3.6. □

Theorem 3.9 *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is regular if and only if one of the following conditions holds:*

(1) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \widetilde{\cap} \langle H, C\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle H, C\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal $\langle F, A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideal $\langle G, B\rangle$, and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle G, B\rangle$ over S;

(2) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \widetilde{\cap} \langle H, C\rangle =_{(\gamma,\delta)} \langle F, A\rangle \odot \langle G, B\rangle \odot \langle H, C\rangle$ for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right ideal $\langle F, A\rangle$, any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft ideal $\langle G, B\rangle$, and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left ideal $\langle G, B\rangle$ over S.

Proof. The proof is similar to that of Theorem 3.8. □

4 Conclusions

As a continuation of the work of Yin et al. [14], this paper focused on the characterization of regular ordered semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft bi-ideals and obtained some new interesting results. Our future work on this topic will focus on studying the characterization of other ordered semigroups.

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