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Multiple solutions for a semilinear nonhomogeneous [1] elliptic system

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Abstract

In this paper, by the Mountain Pass Theory and Ekeland's variational principle, we consider the existence and multiplicity of nontrivial solutions for the nonhomogeneous semilinear elliptic system

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^{\beta} + l_1(x), \ x \in \Omega, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(x)|u|^{\alpha}|v|^{\beta-2}v + l_2(x), \ x \in \Omega, \\ \frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u, \ \frac{\partial v}{\partial n} = \mu h(x)|v|^{q-2}v, \ x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1, \ \beta > 1$ satisfying $2 < \alpha + \beta < 2^*$ $(2^* = \frac{2N}{N-2}$ if $N \geq 3, \ 2^* = \infty$ if $N = 2), \ 1 < q < 2$, the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

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1 Introduction

The aim of this paper is to investigate the following semilinear elliptic system

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^{\beta} + l_1(x),\ x \in \Omega, \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(x)|u|^{\alpha}|v|^{\beta-2}v + l_2(x),\ x \in \Omega, \\ \frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u,\ \frac{\partial v}{\partial n} = \mu h(x)|v|^{q-2}v,\ x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\alpha > 1,\ \beta > 1$ satisfying $2 < \alpha + \beta < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 2$), $1 < q < 2$, the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Existence and multiplicity results for a semilinear elliptic systems with nonlinear boundary condition are widely studied. In [1], Tsung-Fang Wu studied a class of semilinear elliptic equations in $\mathbb{R}^N_+$ with nonlinear boundary condition and sign-changing weight function. By means of the Lusternik-Schnirelman category, multiple positive solutions are obtained. In [2], Hu Li, Xing-Ping Wu and Chun-Lei Tang obtained two positive solutions for a nonlinear homogeneous system with nonlinear homogeneous boundary condition via the Nehari manifold approach. We refer to [3, 4, 5, 6, 7] for additional results on semilinear elliptic problem.

We are motivated by the paper of K.J. Brown and Tsung-Fang Wu [8], in which the equations are homogeneous. We now extend the analysis to the nonhomogeneous equations with nonlinear nonhomogeneous boundary conditions. Replacing the Nehari manifold methods, we will use the Mountain Pass Theory and Ekeland's variational principle to study the existence of multiple solutions for problem (1). It seems difficult to get the same result by Nehari manifold methods.

In order to state our main theorem, let us introduce the following hypotheses:

(H_1) $f \in C(\bar{\Omega})$ with $\|f\|_\infty = 1$ and $f^+ = max\{f, 0\} \neq 0$, $l_1(x), l_2(x) \in L^\sigma(\Omega)$, $\sigma = \frac{2^*}{2^*-1}$. Furthermore, there exists a non-empty open domain $\Omega_1 \subset \Omega$ such that $l_1(x) > 0, l_2(x) > 0$ in Ω_1. (H_2) $g, h \in C(\partial\Omega)$ with $\|g\|_\infty = \|h\|_\infty = 1, g^\pm = max\{\pm g, 0\} \neq 0$ and $h^\pm = max\{\pm h, 0\} \neq 0$.

Throughout this paper, we let S and $\bar{S}$ be the best Sobolev and the best Sobolev trace constants for the embedding of $H^1(\Omega)$ in $L^{2^*}(\Omega)$ and $H^1(\Omega)$ in $L^q(\partial\Omega)$, respectively. Then we have the following result.

Theorem 1.1. *Assume that $(H_1)-(H_2)$ hold, there exist $c_0, c_1 > 0$ such that the problem (1) admits at least two solutions provided $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < c_0$ and $\|l_1\|^2_\sigma + \|l_2\|^2_\sigma < c_1(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{\alpha+\beta-q}}$, where $\sigma = \frac{2^*}{2^*-1}$.*

2 The proof of Theorem 1.1.

In this paper, we let $H = H^1(\Omega) \times H^1(\Omega)$ with the standard norm

$$\|(u,v)\|_H = \Big(\int_\Omega (|\nabla u|^2 + u^2)dx + \int_\Omega (|\nabla v|^2 + v^2)dx \Big)^{1/2}.$$

First we give the definition of the weak solution of (1).

Definition 2.1. *We say that (u, v) is a weak solution to (1) if for all $(\varphi_1, \varphi_2) \in H$ we have*

$$\int_\Omega (\nabla u \nabla \varphi_1 + u\varphi_1)dx + \int_\Omega (\nabla v \nabla \varphi_2 + v\varphi_2)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega f|u|^{\alpha-2}u|v|^\beta \varphi_1 dx$$
$$- \frac{\beta}{\alpha+\beta}\int_\Omega f|u|^\alpha |v|^{\beta-2} v\varphi_2 dx - \int_\Omega l_1(x)\varphi_1 dx - \int_\Omega l_2(x)\varphi_2 dx$$
$$- \lambda \int_{\partial\Omega} g|u|^{q-2}u\varphi_1 ds - \mu \int_{\partial\Omega} h|v|^{q-2}v\varphi_2 ds = 0.$$

Let $J_{\lambda,\mu} : H \to \mathbb{R}^1$ be the energy functional of problem (1) defined by

$$J_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{\alpha+\beta}\int_\Omega f|u|^\alpha|v|^\beta dx - L(u,v) - \frac{1}{q}G_{\lambda,\mu}(u,v), \quad (2)$$

where

$$L(u,v) = \int_\Omega \big(l_1(x)u + l_2(x)v\big)dx, G_{\lambda,\mu}(u,v) = \lambda\int_{\partial\Omega} g|u|^q ds + \mu \int_{\partial\Omega} h|v|^q ds.$$

We see $J_{\lambda,\mu}(u,v) \in C^1(H, \mathbb{R}^1)$ and for any $(\varphi_1, \varphi_2) \in H$ there holds

$$\langle J'_{\lambda,\mu}(u,v), (\varphi_1,\varphi_2)\rangle = \int_\Omega (\nabla u \nabla \varphi_1 + u\varphi_1)dx + \int_\Omega (\nabla v \nabla \varphi_2 + v\varphi_2)dx$$
$$- \frac{\alpha}{\alpha+\beta}\int_\Omega f|u|^{\alpha-2}u|v|^\beta\varphi_1 dx - \frac{\beta}{\alpha+\beta}\int_\Omega f|u|^\alpha|v|^{\beta-2}v\varphi_2 dx$$
$$- \int_\Omega l_1(x)\varphi_1 dx - \int_\Omega l_2(x)\varphi_2 dx - \lambda\int_{\partial\Omega} g|u|^{q-2}u\varphi_1 ds$$
$$- \mu\int_{\partial\Omega} h|v|^{q-2}v\varphi_2 ds. \quad (3)$$

We will make use of the Mountain Pass Theorem in [9] (also see [10]).

Lemma 2.2. *(Mountain Pass Theorem) Suppose X is a Banach space, $I \in C^1(X, \mathbb{R}^1)$ with $I(0) = 0$. If I satisfies (PS) condition and*

(A_1) there are ρ, $\alpha_0 > 0$, such that $I(u) \geq \alpha_0$ when $\|u\|_X = \rho$.

(A_2) *there is* $e \in X,\ \|e\|_X > \rho$ *such that* $I(e) < 0$.
Define

$$\Gamma = \{\gamma \in C^1([0,1], X) | \gamma(0) = 0, \gamma(1) = e\}$$

Then

$$c = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I(\gamma(t)) \ge \alpha_0$$

is a critical value of $I(u)$.

Lemma 2.3. *Assume* $(H_1) - (H_2)$ *hold. Then there exist* $c_0, c_1 > 0$ *such that* $J_{\lambda,\mu}(u,v)$ *satisfies the assumption* $(A_1) - (A_2)$ *in Lemma 2.1 provided* $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < c_0$, *and* $\|l_1\|_\sigma^2 + \|l_2\|_\sigma^2 < c_1(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{\alpha+\beta-q}}$, *where* $\sigma = \frac{2^*}{2^*-1}$.

Proof. By the Hölder inequality and the Sobolev embedding theorem we have

$$\begin{aligned}\int_\Omega f(x)|u|^{\alpha+\beta}dx &\le \Big(\int_\Omega |f(x)|^\delta dx\Big)^{\frac{1}{\delta}} \Big(\int_\Omega |u|^{2^*}dx\Big)^{\frac{\alpha+\beta}{2^*}} \\ &\le M_1 S^{\alpha+\beta}\|u\|_{H^1(\Omega)}^{\alpha+\beta},\end{aligned}$$

where $\delta = \frac{2^*}{2^*-(\alpha+\beta)}$, $M_1 > 0$, such that $(\int_\Omega |f(x)|^\delta dx)^{\frac{1}{\delta}} < M_1$.

In a similar manner we obtain

$$\int_\Omega f(x)|v|^{\alpha+\beta}dx \le M_1 S^{\alpha+\beta}\|v\|_{H^1(\Omega)}^{\alpha+\beta}.$$

So

$$\int_\Omega f(x)|u|^\alpha|v|^\beta dx \le 2M_1 S^{\alpha+\beta}\|(u,v)\|_H^{\alpha+\beta}. \tag{4}$$

It is clear that

$$\begin{aligned}&\lambda\int_{\partial\Omega} g|u|^q ds + \mu\int_{\partial\Omega} h|v|^q ds \\ &\le \lambda\|g\|_\infty \int_{\partial\Omega}|u|^q ds + \mu\|h\|_\infty\int_{\partial\Omega}|v|^q ds \\ &\le \bar{S}^q(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}}\|(u,v)\|_H^q.\end{aligned} \tag{5}$$

Using Young inequality, we get

$$\int_\Omega |l_1(x)||u|dx \le \|l_1\|_\sigma\|u\|_{2^*} \le S\|l_1\|_\sigma\|u\|_{H^1} \le \varepsilon\|(u,v)\|_H^2 + C_\varepsilon\|l_1\|_\sigma^2, \tag{6}$$

$$\int_\Omega |l_2(x)||v|dx \le \|l_2\|_\sigma \|v\|_{2^*} \le S\|l_2\|_\sigma \|v\|_{H^1} \le \varepsilon\|(u,v)\|_H^2 + C_\varepsilon \|l_2\|_\sigma^2, \tag{7}$$

where $\sigma = \frac{2^*}{2^*-1},\ \varepsilon > 0,\ C_\varepsilon > 0$. Hence for $0 < \varepsilon \le \frac{1}{8}$, we have

$$\begin{aligned} J_{\lambda,\mu}(u,v) \ge& \frac{1}{2}\|(u,v)\|_H^2 - \frac{2}{\alpha+\beta} M_1 S^{\alpha+\beta} \|(u,v)\|_H^{\alpha+\beta} \\ &- \frac{1}{q}\bar{S}^q (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u,v)\|_H^q \\ &- 2\varepsilon \|(u,v)\|_H^2 - C_\varepsilon \|l_1\|_\sigma^2 - C_\varepsilon \|l_2\|_\sigma^2 \\ \ge& \frac{1}{4}\|(u,v)\|_H^2 - \frac{2}{\alpha+\beta} M_1 S^{\alpha+\beta} \|(u,v)\|_H^{\alpha+\beta} \\ &- \frac{1}{q}\bar{S}^q (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}} \|(u,v)\|_H^q - C_\varepsilon \|l_1\|_\sigma^2 - C_\varepsilon \|l_2\|_\sigma^2. \end{aligned} \tag{8}$$

Let $g(z) := \frac{2}{\alpha+\beta} M_1 S^{\alpha+\beta} z^{\alpha+\beta-2} + \frac{1}{q}\bar{S}^q (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}} z^{q-2},\ z > 0$.

To verify (A_1) in Lemma 2.1, it suffices to show that $g(z_1) < \frac{1}{4}$ for some $z_1 = \|(u,v)\|_H > 0$. Note that $g(z) \to +\infty$ when $z \to 0^+$ or $z \to +\infty$, $g(z)$ has a minimum at $z_1 > 0$. Then $g'(z_1) = 0$. A simple computation yields

$$z_1 = \left(\frac{\bar{S}^q (\alpha+\beta)(2-q)}{2qM_1(\alpha+\beta-2)S^{\alpha+\beta}} \right)^{\frac{1}{\alpha+\beta-q}} (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2(\alpha+\beta-q)}}. \tag{9}$$

Moveover, $g(z_1) < \frac{1}{4}$ implies that

$$\begin{aligned} &\frac{2}{\alpha+\beta} M_1 S^{\alpha+\beta} \left(\frac{\bar{S}^q (\alpha+\beta)(2-q)}{2qM_1(\alpha+\beta-2)S^{\alpha+\beta}} \right)^{\frac{\alpha+\beta-2}{\alpha+\beta-q}} (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{(2-q)(\alpha+\beta-2)}{2(\alpha+\beta-q)}} \\ &+ \frac{1}{q}\bar{S}^q \left(\frac{\bar{S}^q (\alpha+\beta)(2-q)}{2qM_1(\alpha+\beta-2)S^{\alpha+\beta}} \right)^{\frac{q-2}{\alpha+\beta-q}} (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{(2-q)(\alpha+\beta-2)}{2(\alpha+\beta-q)}} < \frac{1}{4}. \end{aligned} \tag{10}$$

Let

$$\begin{aligned} &\frac{1}{4}z_1^2 - \frac{2}{\alpha+\beta} M_1 S^{\alpha+\beta} z_1^{\alpha+\beta} - \frac{1}{q}\bar{S}^q (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}} z_1^q \\ &- C_\varepsilon (\|l_1\|_\sigma^2 + \|l_2\|_\sigma^2) > 0, \end{aligned} \tag{11}$$

we deduce that $\|l_1\|_\sigma^2 + \|l_2\|_\sigma^2 < c_1 (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{\alpha+\beta-q}}$, where the constant $c_1 > 0$ is independent of λ, μ. Then it follows from (8) , (10) and (11) that there exist $c_0,\ \alpha_0 > 0$ such that $J_{\lambda,\mu}(u,v) \ge \alpha_0$ with $0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < c_0$ and $\|l_1\|_\sigma^2 + \|l_2\|_\sigma^2 < c_1 (|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{\alpha+\beta-q}}$. Thus (A_1) in Lemma 2.1 is true.

We now verify (A_2) in Lemma 2.1. Choose $(\varphi_1, \varphi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, $(\varphi_1, \varphi_2) \neq 0$. Then

$$J_{\lambda,\mu}(t\varphi_1, t\varphi_2) = \frac{1}{2}t^2\|(\varphi_1, \varphi_2)\|_H^2 - \frac{1}{\alpha+\beta}t^{\alpha+\beta}\int_\Omega f|\varphi_1|^\alpha|\varphi_2|^\beta dx$$
$$- t(\int_\Omega l_1(x)\varphi_1 dx - \int_\Omega l_2(x)\varphi_2 dx) - \frac{1}{q}t^q(\lambda\int_{\partial\Omega} g|\varphi_1|^q ds + \mu\int_{\partial\Omega} h|\varphi_2|^q ds)$$

and $J_{\lambda,\mu}(t\varphi_1, t\varphi_2) \to -\infty$ as $t \to +\infty$ since $\alpha + \beta > 2$. Therefore, there exists t large enough, such that $J_{\lambda,\mu}(t\varphi_1, t\varphi_2) < 0$. Then, we take $e = (t\varphi_1, t\varphi_2) \in H$ and $J_{\lambda,\mu}(e) < 0$ and (A_2) in Lemma 2.1 is true. This completes the proof of Lemma 2.2. □

Lemma 2.4. *Assume $(H_1) - (H_2)$ hold. Then $J_{\lambda,\mu}(u, v)$ defined by (2) satisfies (PS) condition on H.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$ sequence of $J_{\lambda,\mu}(u, v)$ in H, that is

$$J_{\lambda,\mu}(u_n, v_n) \to c, \ J'_{\lambda,\mu}(u_n, v_n) \to 0 \text{ in } H^*.$$

We first claim that $\{(u_n, v_n)\}$ is bounded in H. In fact, for large n we obtain

$$\begin{aligned}
&c + 1 + \|(u_n, v_n)\|_H \\
&\geq J_{\lambda,\mu}(u_n, v_n) - \frac{1}{\alpha+\beta}\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\
&\geq (\frac{1}{2} - \frac{1}{\alpha+\beta})\|(u_n, v_n)\|_H^2 + (\frac{1}{\alpha+\beta} - \frac{1}{q})\bar{S}^q(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}})^{\frac{2-q}{2}}\|(u_n, v_n)\|_H^q \\
&+ (\frac{1}{\alpha+\beta} - 1)S(\|l_1\|_\sigma + \|l_2\|_\sigma)\|(u_n, v_n)\|_H. \qquad (12)
\end{aligned}$$

By virtue of (H_2) and (12), we conclude that $\{\|(u_n, v_n)\|_H\}$ is bounded. Thus, passing to a subsequence, if necessary, we have $\|(u_n, v_n)\|_H \to t_0 \geq 0$.

If $t_0 = 0$, then the proof is finished. In the following, we now show that $\{(u_n, v_n)\}$ has a convergent subsequence in H for $t_0 > 0$. Since $\{(u_n, v_n)\}$ is bounded in H, the $\{u_n\}$ and $\{v_n\}$ are bounded in $H^1(\Omega)$ respectively. Then, there exists $\{(u, v)\} \in H$ such that

$$u_n \rightharpoonup u, \ v_n \rightharpoonup v \text{ weakly in } H^1(\Omega).$$

Let

$$
\begin{aligned}
P_n &:= \langle J'_{\lambda,\mu}(u_n, v_n), (u_n - u, v_n - v)\rangle \\
&= \int_\Omega (\nabla u_n \nabla(u_n - u) + u_n(u_n - u))dx \\
&+ \int_\Omega (\nabla v_n \nabla(v_n - v) + v_n(v_n - v))dx \\
&- \frac{\alpha}{\alpha+\beta}\int_\Omega f|u_n|^{\alpha-2}u_n|v_n|^\beta(u_n - u)dx \\
&- \frac{\beta}{\alpha+\beta}\int_\Omega f|u_n|^{\alpha}|v_n|^{\beta-2}v_n(v_n - v)dx \\
&- \int_\Omega l_1(x)(u_n - u)dx - \int_\Omega l_2(x)(v_n - v)dx \\
&- \lambda\int_{\partial\Omega} g|u_n|^{\alpha-2}u_n(u_n - u)ds - \mu\int_{\partial\Omega} h|v_n|^{q-2}v_n(v_n - v)ds. \qquad (13)
\end{aligned}
$$

Then the fact $J'_{\lambda,\mu}(u_n, v_n) \to 0$ in H^* implies $P_n \to 0$ as $n \to \infty$.
Since that $u_n \rightharpoonup u,\ v_n \rightharpoonup v$ in $H^1(\Omega)$, we see that

$$
\begin{aligned}
Q_n &:= \int_\Omega (\nabla u \nabla(u_n - u) + u(u_n - u))dx \\
&+ \int_\Omega (\nabla v \nabla(v_n - v) + v(v_n - v))dx \to 0,\ n \to \infty.
\end{aligned}
$$

From (H_1) and Dominated Convergence Theory and the Sobolev compact embedding theory, we can conclude

$$\int_\Omega f|u_n|^{\alpha-2}u_n|v_n|^\beta(u_n - u)dx \to 0,\ \text{as } n \to \infty, \qquad (14)$$

$$\int_\Omega f|u_n|^{\alpha}|v_n|^{\beta-2}v_n(v_n - v)dx \to 0,\ \text{as } n \to \infty. \qquad (15)$$

By the Sobolev trace embedding theory in the bounded domain Ω, $\{u_n\}$ has a subsequence, still denoted by $\{u_n\}$, which converges u in $L^q(\partial\Omega)$. Thus we have

$$\int_{\partial\Omega} |g||u_n - u|^q ds \to 0,\ as\ n \to \infty.$$

Then it follows from *Hölder* inequality that as $n \to \infty$,

$$
\begin{aligned}
\int_{\partial\Omega} |g||u_n|^{q-1}|u_n - u|ds &\le \Big(\int_{\partial\Omega} |g||u_n - u|^q ds\Big)^{\frac{1}{q}}\Big(\int_{\partial\Omega} |g||u_n|^q ds\Big)^{\frac{q-1}{q}} \\
&\to 0,\ as\ n \to \infty. \qquad (16)
\end{aligned}
$$

Similarly, we have

$$\int_{\partial\Omega} |h||v_n|^{q-1}|v_n - v|ds \to 0, \text{ as } n \to \infty, \tag{17}$$

$$\int_{\Omega} |l_1(x)||u_n - u|dx \to 0, \text{ as } n \to \infty, \tag{18}$$

$$\int_{\Omega} |l_2(x)||v_n - v|dx \to 0, \text{ as } n \to \infty. \tag{19}$$

Then it follows from (13) and (14)-(19) that as $n \to \infty$,

$$\begin{aligned} T_n := & \int_{\Omega} (\nabla u_n \nabla (u_n - u) + u_n(u_n - u))dx \\ & + \int_{\Omega} (\nabla v_n \nabla (v_n - v) + v_n(v_n - v))dx \to 0. \end{aligned}$$

Thus $T_n - Q_n \to 0$. That is to say

$$T_n - Q_n = \|(u_n - u, v_n - v)\|_H^2 \to 0, \text{ as } n \to \infty. \tag{20}$$

Thus $J_{\lambda,\mu}(u, v)$ satisfies $(PS)_c$ condition on H. This completes the proof of Lemma 2.3.

Proof of Theorem 1.1. By Lemma 2.2 and Lemma 2.3, $J_{\lambda,\mu}(u, v)$ satisfies all assumptions in Lemma 2.1. Then there exists $(u_1, v_1) \in H$ such that (u_1, v_1) is a solution of problem (1) by Lemma 2.1. Furthermore, $J_{\lambda,\mu}(u, v) \geq \alpha_0 > 0$.

We now seek a solution (u_2, v_2) of problem (1). Choose $(\varphi_1, \varphi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ such that $\int_\Omega l_1(x)\varphi_1 dx + \int_\Omega l_2(x)\varphi_2 dx > 0$ and then

$$\begin{aligned} J_{\lambda,\mu}(t\varphi_1, t\varphi_2) = & \frac{1}{2}t^2\|(\varphi_1, \varphi_2)\|_H^2 - \frac{1}{\alpha+\beta}t^{\alpha+\beta} \int_\Omega f|\varphi_1|^\alpha|\varphi_2|^\beta dx \\ & - t(\int_\Omega l_1(x)\varphi_1 dx + \int_\Omega l_2(x)\varphi_2 dx) \\ & - \frac{1}{q}t^q(\lambda \int_{\partial\Omega} g|\varphi_1|^q ds + \mu \int_{\partial\Omega} h|\varphi_2|^q ds) \\ & < 0 \end{aligned} \tag{21}$$

for small $t > 0$ and thus for any open ball $B_\tau \subset H$ such that

$$-\infty < c_\tau = \inf_{\overline{B}_\tau} J_{\lambda,\mu}(u, v) < 0. \tag{22}$$

Thus, exists $\rho > 0$, such that

$$c_\rho = \inf_{(u,v)\in\overline{B}_\rho} J_{\lambda,\mu}(u, v) < 0 \quad \text{and} \quad \inf_{(u,v)\in\partial B_\rho} J_{\lambda,\mu}(u, v) > 0. \tag{23}$$

Letting $\varepsilon_n \downarrow 0$ such that

$$0 < \varepsilon_n < \inf_{(u,v)\in\partial B_\rho} J_{\lambda,\mu}(u,v) - \inf_{(u,v)\in B_\rho} J_{\lambda,\mu}(u,v). \tag{24}$$

Then, by Ekeland's variational principle in [11], there exists $\{(u_n, v_n)\} \subset \overline{B}_\rho$ such that

$$c_\rho \leq J_{\lambda,\mu}(u_n, v_n) < c_\rho + \varepsilon_n \tag{25}$$

and

$$J_{\lambda,\mu}(u_n, v_n) < J_{\lambda,\mu}(u,v) + \varepsilon_n \|(u_n - u, v_n - v)\|_H, \tag{26}$$

for all $(u,v) \in \overline{B}_\rho,\ u \neq u_n,\ v \neq v_n$.

Then it follows from (24)-(26) that

$$J_{\lambda,\mu}(u_n, v_n) < c_\rho + \varepsilon_n \leq \inf_{(u,v)\in B_\rho} J_{\lambda,\mu}(u,v) + \varepsilon_n < \inf_{(u,v)\in\partial B_\rho} J_{\lambda,\mu}(u,v), \tag{27}$$

so that $(u_n, v_n) \in B_\rho$. We now consider the functional $F : \overline{B}_\rho \to \mathbb{R}$ given by

$$F(u,v) = J_{\lambda,\mu}(u,v) + \varepsilon_n \|(u - u_n, v - v_n)\|_H, \quad (u,v) \in \overline{B}_\rho. \tag{28}$$

Then (26) shows that $F(u_n, v_n) < F(u,v), (u,v) \in \overline{B}_\rho,\ u \neq u_n, v \neq v_n$ and thus (u_n, v_n) is a strict local minimum of F. Moreover,

$$t^{-1}(F(u_n + t\varphi_1, v_n + t\varphi_2) - F(u_n, v_n)) \geq 0, \tag{29}$$

for small $t > 0$ and $(\varphi_1, \varphi_2) \in B_1$. Hence,

$$t^{-1}(J_{\lambda,\mu}(u_n + t\varphi_1, v_n + t\varphi_2) - J_{\lambda,\mu}(u_n, v_n)) + \varepsilon_n \|(\varphi_1, \varphi_2)\|_H \geq 0. \tag{30}$$

Let $t \to 0^+$, then

$$\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2)\rangle + \varepsilon_n \|(\varphi_1, \varphi_2)\|_H \geq 0, \quad \forall (\varphi_1, \varphi_2) \in B_1. \tag{31}$$

Replacing (φ_1, φ_2) in (31) by $(-\varphi_1, -\varphi_2)$, we get

$$-\langle J'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2)\rangle + \varepsilon_n \|(\varphi_1, \varphi_2)\|_H \geq 0, \quad \forall (\varphi_1, \varphi_2) \in B_1. \tag{32}$$

So that $\|J'_{\lambda,\mu}(u_n, v_n)\| \leq \varepsilon_n$.
Therefore, there is a sequence $\{(u_n, v_n)\} \subset B_\rho$ such that $J_{\lambda,\mu}(u_n, v_n) \to c_\rho < 0$, and
$J'_{\lambda,\mu}(u_n, v_n) \to 0$ in H^* as $n \to \infty$. By Lemma 2.3, $\{(u_n, v_n)\}$ has a convergent subsequence in H, still denoted by $\{(u_n, v_n)\}$, such that $(u_n, v_n) \to (u_2, v_2)$ in H. Thus (u_2, v_2) is a solution of (1) with $J_{\lambda,\mu}(u_2, v_2) < 0$. Then the proof of Theorem 1.1 is complete. □

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