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Moderate deviations for one-dimensional random walk in random scenery

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Abstract

In this paper, we investigate the moderate deviations for one dimensional random walks in independent, identically distributed random sceneries. Our approach is based on the Gätner-Ellis theorem. As an application, we get the corresponding law of the iterated logarithm.

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1 Introduction

In 1979, Kesten and Spitzer ([12]) introduced a different model for random walk in random environment, which they call random walk in random scenery. In the field of stochastic processes in random environments, random walks in random scenery represent a class of processes with fairly weak interaction. Recently, they have received a lot of attention.

To define random walk in random scenery, suppose $\{S_n : n \geq 0\}$ is an underlying random walk on $\mathbb{Z}$ started at the origin, and $\{\xi(i) : i \in \mathbb{Z}\}$ are independent, identically distributed real-valued random variables, which are independent of the random walk and which are called the scenery. Random walk in random scenery is the process $\{X_n : n \geq 0\}$ given by

$$X_n := \sum_{1\leq k\leq n} \xi(S_k) = \sum_{x\in\mathbb{Z}} \xi(x) l_n(x),\ \ for\ n \geq 0,$$

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where $l_n(x) = \sum_{1\le k\le n} 1_{\{S_k=x\}}$ are the local times of the random walk at the site x.

In the early papers ([4], [12]), the authors established central limit theorems for the random walk in random scenery. Large deviation problems for random walks in random scenery have only recently attracted attention, see ([2], [3], [9], [10], [11]), and also ([1], [6]) where Brownian motions are used in place of random walks. Recently, the authors of [9] have investigated moderate deviation principles for X_n in dimension $d \ge 2$, providing a full analysis including explicit rate functions. Crucial ingredients of their proofs are concentration inequalities for self-intersection local times of random walks.

In this paper, we study the moderate deviations for X_n in dimension $d = 1$. In the rest of this paper, $\{S_n : n \ge 0\}$ is a symmetric random walk on $\mathbb{Z}$ with covariance σ^2. We assume that the smallest group that supports $\{S_n : n \ge 0\}$ is $\mathbb{Z}$. Throughout, $\{\xi(i) : i \in \mathbb{Z}\}$ is an i.i.d. sequence of symmetric random variable satisfying

$$\mathbb{E}\xi(1)^2 = 1 \qquad and \qquad \mathbb{E}e^{\lambda_0 \xi(1)^2} < \infty, \ for\ some\ \lambda_0 > 0.$$

Our main approach is based on high moment estimations and Gätner-Ellis theorem. Some ideas of the proof come from [7]. The main result is the following theorem.

Theorem 1.1. *Let b_n be a positive sequence satisfying*

$$b_n \to \infty, \ b_n = o(\sqrt[7]{n}), \ n \to \infty. \tag{1.1}$$

Then, for any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left(\pm X_n \ge \lambda (n b_n)^{3/4}\right) = -\sqrt[3]{\frac{81}{32}}\sigma^{2/3}\lambda^{4/3}. \tag{1.2}$$

As an application, we get the following law of the iterated logarithm.

Corollary 1.1.

$$\limsup_{n\to\infty} \frac{\pm X_n}{(2n\log\log n)^{3/4}} = \frac{\sqrt{2}}{3}\sigma^{-1/2}, \ a.s..$$

2 Proof of Theorem1.1

We define

$$H_n = \sum_{x\in\mathbb{Z}} l_n^2(x).$$

Recall $l_n(x) = \sum_{1\le k\le n} 1_{\{S_k=x\}}$ are the local times of the random walk at the site x. Let K_n be a positive sequence will later be specified.

The following two random quantities play important roles in this paper:

$$\tilde{X}_n = X_n I_{\left\{\sup\limits_{x\in\mathbb{Z}} l_n(x)\le K_n\right\}},$$

$$\tilde{H}_n = H_n I_{\left\{\sup\limits_{x\in\mathbb{Z}} l_n(x)\le K_n\right\}}.$$

Firstly, we give two useful Lemmas.

Lemma 2.1. *(see [8]) Set* $Q_n = \sum\limits_{1\le j<k\le n} I_{\{S_j=S_k\}}$, *and* b_n *is a positive sequence satisfying (1.1). Then, for any* $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left(Q_n \ge \lambda n^{3/2} b_n^{1/2}\right) = -6\sigma^2\lambda^2. \tag{2.1}$$

Lemma 2.2. *(see [8]) set* $K_n = M_n\sqrt{nb_n}$ *where* M_n *satisfying*

$$M_n \to \infty,\ M_n^2 \left(\frac{b_n^7}{n}\right)^{\frac{1}{4}} \to 0,\ n\to\infty. \tag{2.2}$$

and b_n *is a positive sequence satisfying (1.1). Then,*

$$\lim_{n\to\infty} \log \mathbb{P}\left(\sup_{x\in\mathbb{Z}} l_n(x) > K_n\right) = -\infty. \tag{2.3}$$

We now prove the moderate deviations for $\tilde{X}_n$.

Proposition 2.1. *Let* b_n *be a positive sequence satisfying*

$$b_n \to \infty,\ b_n = o(\sqrt[7]{n}),\ n\to\infty.$$

Then, for any $\theta > 0$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{E}\exp\left\{\pm\theta \frac{b_n^{1/4}}{n^{3/4}} \tilde{X}_n\right\} = \frac{\theta^4}{24\sigma^2}. \tag{2.4}$$

Proof 2.1. *In view of* $Q_n = \frac{1}{2}(H_n - n)$, *we have*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\left(H_n \ge \lambda n^{3/2} b_n^{1/2}\right) = -\frac{3}{2}\sigma^2\lambda^2.$$

Under the fact that $H_n \le n \sup_{x\in\mathbb{Z}} l_n(x)$, *we have for any* $\lambda > 0$,

$$\mathbb{E}\exp\left\{\lambda \frac{b_n^{1/2}}{n^{3/2}} H_n\right\} \le \mathbb{E}\exp\left\{\lambda\sqrt{\frac{b_n}{n}} \sup_{x\in\mathbb{Z}} l_n(x)\right\}.$$

By the fact (see Lemma 11 and Lemma 12 in [13]) that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{\lambda}{2}\sqrt{\frac{b_n}{n}}\sup_{x\in\mathbb{Z}}l_n(x)\right\}<\infty,$$

we have that for any $\lambda>0$*,*

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\lambda\frac{b_n^{1/2}}{n^{3/2}}H_n\right\}<\infty.$$

According to Varadhan's integral lemma, we have for any $\lambda>0$*,*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\lambda\frac{b_n^{1/2}}{n^{3/2}}H_n\right\}=\sup_{y>0}\{y\lambda-\frac{3}{2}\sigma^2y^2\}=\frac{\lambda^2}{6\sigma^2} \tag{2.5}$$

We now compute the Laplace transform $\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{X}_n\right\}$. *Integrating with respect to randomness of the i.i.d scenery* $\{\xi(i):i\in\mathbb{Z}\}$*, lead to*

$$\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{X}_n\right\}=\mathbb{E}\exp\left\{\sum_{x\in\mathbb{Z}}\frac{\theta^2b_n^{1/2}}{2n^{3/2}}l_n(x)I_{\left\{\sup\limits_{x\in\mathbb{Z}}\le K_n\right\}}(1+o(1))\right\}$$

where Λ *is the log-Laplace transform of the variables* $\{\xi(i):i\in\mathbb{Z}\}$*. Since only the behavior of* Λ *near the origin is concerned. Using the strong moments assumptions of* $\{\xi(i):i\in\mathbb{Z}\}$ *, we have* $\Lambda(\theta)=\frac{\theta^2}{2}(1+o(1))$*. Therefore,*

$$\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{X}_n\right\}=\mathbb{E}\exp\left\{\frac{\theta^2b_n^{1/2}}{2n^{3/2}}\tilde{H}_n(1+o(1))\right\}.$$

By (2.5), we have

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{X}_n\right\}\le\frac{\theta^4}{24\sigma^2}. \tag{2.6}$$

Notice that

$$\begin{aligned}\mathbb{E}\exp\left\{\frac{\theta^2b_n^{1/2}}{2n^{3/2}}\tilde{H}_n\right\}&\ge\mathbb{E}\left[\exp\left\{\frac{\theta^2b_n^{1/2}}{2n^{3/2}}H_n\right\}I_{\left\{\sup\limits_{x\in\mathbb{Z}}l_n(x)\le K_n\right\}}\right]\\&=\mathbb{E}\exp\left\{\frac{\theta^2b_n^{1/2}}{2n^{3/2}}H_n\right\}-\mathbb{E}\left[\exp\left\{\frac{\theta^2b_n^{1/2}}{2n^{3/2}}H_n\right\}I_{\left\{\sup\limits_{x\in\mathbb{Z}}l_n(x)>K_n\right\}}\right].\end{aligned}$$

In view of (2.5),

$$\max\left\{\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\frac{\theta^2 b_n^{1/2}}{2n^{3/2}}\tilde{H}_n\right\},\right.$$
$$\left.\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\frac{\theta^2 b_n^{1/2}}{2n^{3/2}}H_n\right\}I_{\left\{\sup_{x\in\mathbb{Z}} l_n(x)>K_n\right\}}\right]\right\}\geq\frac{\theta^4}{24\sigma^2}.$$

By Lemma2.2 and Cauchy-Schwartz inequality and (2.5), we have

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\left[\exp\left\{\frac{\theta^2 b_n^{1/2}}{2n^{3/2}}H_n\right\}I_{\left\{\sup_{x\in\mathbb{Z}} l_n(x)>K_n\right\}}\right]=-\infty,$$

which implies that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{H}_n\right\}\geq\frac{\theta^4}{24\sigma^2}. \tag{2.7}$$

So,

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\left\{\pm\theta\frac{b_n^{1/4}}{n^{3/4}}\tilde{X}_n\right\}\geq\frac{\theta^4}{24\sigma^2}. \tag{2.8}$$

Proposition2.1 follows from (2.6) and (2.8).

We now complete the proof of Theorem1.1.

By G*ä*tner-Ellis theorem, we have for any $\lambda>0$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\left(\pm\tilde{X}_n\geq\lambda(nb_n)^{3/4}\right)=-\sup_{\theta>0}\left\{\lambda\theta-\frac{\theta^4}{24\sigma^2}\right\}=-\sqrt[3]{\frac{81}{32}}\sigma^{2/3}\lambda^{4/3}.$$

Then the moderate deviations for X_n can be obtained through the following exponential equivalence given by

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}(\tilde{X}_n\neq X_n)=\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\left(\sup_{x\in\mathbb{Z}} l_n(x)>K_n\right)=-\infty.$$

As an application of Theorem1.1, by the standard Borel-Cantelli lemma argument, we can easily get Corallory1.1.

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