Local Regularity for Very Weak Solutions to Elliptic Equations with Degenerate Coercivity

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Abstract
This paper deals with very weak solutions to elliptic equations

\[-\text{div} (a(x,u)Du) = -\text{div} F, \ x \in \Omega,\]

with degenerate coercivity. A local regularity result is obtained under appropriate assumptions.

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1 Introduction and Statement of Result.
Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N, N \geq 2 \). Consider the elliptic equation

\[-\text{div}(a(x,u)Du) = -\text{div} F, \ x \in \Omega, \quad (1.1)\]

where \( a(x, s) : \Omega \times \mathbb{R} \to \mathbb{R} \) is a measurable Carathéodory function satisfying

\[\frac{\alpha}{(1+|s|)^\theta} \leq a(x, s) \leq \frac{\beta}{(1+|s|)^\theta}, \ 0 < \theta < 1, \quad (1.2)\]

where \( 0 < \alpha \leq \beta < \infty \).
Definition 1.1 A function \( u \in W^{1,r}_{loc}(\Omega) \), \( 1 < r < 2 \), is called a very weak solution to (1.1) if
\[
\int_{\Omega} a(x, u) Du D\varphi dx = \int_{\Omega} F D\varphi dx
\]
holds true for any \( \varphi \in W^{1,\frac{r}{r-1}}(\Omega) \) with compact support.

This paper considers local regularity property for very weak solutions to (1.1) with \( a(x, s) \) satisfying (1.2). Local regularity theory is important among the regularity theories of elliptic PDEs. Some local regularity results can be found in the literature [1-6].

The main result of this paper is the following theorem.

Theorem 1.2 Let \( F \in L^m_{loc}(\Omega) \), \( r < m < N \). There exists a constant \( \varepsilon_0 = \varepsilon_0(\alpha, \beta, N) \) such that every very weak solution \( u \in W^{1,r}_{loc}(\Omega) \) with \( r \geq 2 - \varepsilon_0 \) is actually in \( L^{m^*}_{loc}(1-\theta)(\Omega) \), where \( m^* = \frac{Nm}{N-m} \).

In order to prove the above theorem, we need two preliminary lemmas. The first lemma can be found in [5].

Lemma 1.3 Let \( f(\tau) \) be a non-negative bounded function defined for \( 0 \leq R_0 \leq t \leq R_1 \). Suppose that for all \( B_{R_1} \subset \subset \Omega \) the following integral estimate holds
\[
\int_{A_k,\rho} |Dv|^r dx \leq c_1 \left[ \int_{A_{k,R}} \phi_0 dx + (R - \rho)^{-\lambda} \int_{A_{k,R}} |v|^r dx \right],
\]
for every \( k \in \mathbb{N} \) and \( R_0 \leq \rho < R \leq R_1 \), where \( A_{k,R} = B_1 \cap \{|v| > k\} \).
Here \( c_1 = c_1(N, r, m, R_0, R_1, |\Omega|) \), and \( \lambda \) is a positive constant. Then we have \( v \in L^{m^*}_{loc}(\Omega) \).

The following lemma comes from [1].

Lemma 1.4 Let \( v \in W^{1,r}_{loc}(\Omega) \), \( \phi_0 \in L^m_{loc}(\Omega) \), where \( 1 < r < N \) and \( m \) satisfies
\[
r < m < N.
\]
Assume that for all \( B_{R_1} \subset \subset \Omega \) the following integral estimate holds
\[
\int_{A_{k,\rho}} |Dv|^r dx \leq c_1 \left[ \int_{A_{k,R}} \phi_0 dx + (R - \rho)^{-\lambda} \int_{A_{k,R}} |v|^r dx \right],
\]
for every \( k \in \mathbb{N} \) and \( R_0 \leq \rho < R \leq R_1 \), where \( A_{k,R} = B_1 \cap \{|v| > k\} \).
Here \( c_1 = c_1(N, r, m, R_0, R_1, |\Omega|) \), and \( \lambda \) is a positive constant. Then we have \( v \in L^{m^*}_{loc}(\Omega) \).
2 Proof of Theorem 1.2.

In the following, the symbol $C(\ast, \cdots, \ast)$ will denote a constant depends only on the quantities $\ast, \cdots, \ast$, it value may vary from line to line. Define

$$v(x) = \frac{1}{1-\theta} \text{sign}(u)((1 + |u|)^{1-\theta} - 1).$$

It is easy to see that

$$|Du| = \frac{|Du|}{(1 + |u|)^\theta}. \quad (2.1)$$

Let $0 < R_0 < R_1$ be such that $B_{R_0} \subset \subset \Omega$, and $R_0 < \tau < t < R_1$ be arbitrarily fixed. Let $\eta \in C_0^\infty(B_\tau)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_\tau$ and $|D\eta| \leq C(t - \tau)^{-1}$. For any $k \in \mathbb{N}$, we let $T_k(v)$ be the usual truncation of $v$ at level $k$, that is,

$$T_k(v) = \min\{k, \max\{v, -k\}\}.$$ 

We introduce the Hodge decomposition of the vector field $|D(\eta(v - T_k(v)))|^{r-2}$ $D(\eta(v - T_k(v))) \in L^{r/(r-1)}(\Omega)$. Accordingly,

$$|D(\eta(v - T_k(v)))|^{r-2} D(\eta(v - T_k(v))) = D\varphi + h, \quad (2.2)$$

with $\varphi \in W_0^{1,r/(r-1)}(B_t)$ and $h$ a divergence free vector field of class $L^{r/(r-1)}(\Omega, \mathbb{R}^n)$. The reader is referred to [7,8] for estimates concerning such decomposition. We have

$$\|D\varphi\|_{r-1} \leq C(n)\|D(\eta(v - T_k(v)))\|_{r}^{-1} \quad (2.3)$$

and

$$\|h\|_{r-1} \leq C(n)|r - 2\|D(\eta(v - T_k(v)))\|_{r}^{-1}. \quad (2.4)$$

Let

$$E = |D(\eta(v - T_k(v)))|^{r-2} D(\eta(v - T_k(v))) - |\eta D(v - T_k(v))|^{r-2}\eta D(v - T_k(v)).$$

By an elementary inequality (see [9])

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \leq \frac{2^\varepsilon(1 + \varepsilon)}{1 - \varepsilon}|X - Y|^{1-\varepsilon}, \quad X, Y \in \mathbb{R}, \quad 0 < \varepsilon < 1,$$

one has

$$|E| \leq \frac{2^{2-r}(3-r)}{r-1}|(v - T_k(v))D\eta|^{r-1}. \quad (2.5)$$

Take $\varphi$ in the Hodge decomposition (2.2) as a test function in (1.3) we arrive at

$$\int_{A_{k,t}} a(x, u)Du|\eta D(v - T_k(v))|^{r-2}\eta D(v - T_k(v))dx$$

$$= -\int_{A_{k,t}} a(x, u)Dudx + \int_{A_{k,t}} a(x, u)Duhdx + \int_{A_{k,t}} FD\varphi dx \quad (2.6)$$

$$= I_1 + I_2 + I_3.$$
Using (1.2), and noticing (2.1), the left-hand side of the above inequality can be estimated as

$$
\int_{A_{k,t}} a(x,u) Du |\eta D(v - T_k(v))| r^{-2} \eta D(v - T_k(v)) dx
\geq \alpha \int_{A_{k,r}} |Du| (1 + |u|)^\beta |\eta Dv| r^{-2} \eta Dv dx \geq \alpha \int_{A_{k,r}} |Dv|^r dx.
$$

(2.7)

Using (1.2), (2.1), and noticing (2.1), the left-hand side of the above inequality can be estimated as

$$
|I_1| = \left| - \int_{A_{k,t}} a(x,u) Du Edx \right| \leq \beta \int_{A_{k,t}} \frac{|Du|}{(1 + |u|)^\beta} |E| dx
= \beta \int_{A_{k,t}} |Dv||E| dx \leq \beta \|Dv\|_r \|E\|_{\frac{r}{r-1}}
\leq \beta \frac{2^{r-1} - (3 - r)}{r - 1} \|Dv\|_r \|(v - T_k(v)) D\eta\|_{\frac{r}{r-1}}
\leq \beta \frac{2^{r-1} - (3 - r)}{r - 1} \left[ \varepsilon \|Dv\|_r + C(\varepsilon) \|(v - T_k(v)) D\eta\|_{\frac{r}{r-1}} \right]
\leq \beta \frac{2^{r-1} - (3 - r)}{r - 1} \left[ \varepsilon \|Dv\|_r + \frac{C(\varepsilon)}{(t - \tau)^r} \|v\|_{\frac{r}{r-1}} \right],
$$

(2.8)

where \( \| \cdot \|_r = \| \cdot \|_{r,A_{k,t}} \), and have used the fact \(|v - T_k(v)| \leq |v|\).

Using (1.2) and (2.1) again and the estimate (2.4), \(|I_2|\) can be estimated as

$$
|I_2| = \left| \int_{A_{k,t}} a(x,u) Dh dx \right| \leq \beta \int_{A_{k,t}} \frac{|Du|}{(1 + |u|)^\beta} |h| dx
= \beta \int_{A_{k,t}} |Dv||h| dx \leq \beta \|Dv\|_r \|h\|_{\frac{r}{r-1}}
\leq C\beta |r - 2||Dv\|_r \|D(\eta(v - T_k(v)))\|_{\frac{r}{r-1}}
$$

Since

$$
\|D(\eta(v - T_k(v)))\|_r = \|\eta Dv + (v - T_k(v)) D\eta\|_r \leq \|Dv\|_r + \frac{C}{t - \tau} \|v\|_r,
$$

(2.9)

then Young inequality yields

$$
|I_2| \leq C(\varepsilon) \beta |r - 2||Dv\|_r^r + \varepsilon \|Dv\|_r^r + \frac{C}{(t - \tau)^r} \|v\|_r^r.
$$

(2.10)

(2.3), (2.9) together with Young inequality yield

$$
\begin{align*}
|I_3| &= \left| \int_{A_{k,t}} FD\varphi dx \right| \leq \|F\|_r \|D\varphi\|_{\frac{r}{r-1}}
\leq C\|F\|_r \|D(\eta(v - T_k(v)))\|_{\frac{r}{r-1}}
\leq C(\varepsilon) \|F\|_r^r + \varepsilon \|Dv\|_r^r + \frac{C}{(t - \tau)^r} \|v\|_r^r.
\end{align*}
$$

(2.11)
Combining (2.6)-(2.8), (2.10)-(2.11) we arrive at

\[
\alpha \int_{A_{k,\tau}} |Dv|^r dx \leq \left[ \beta \frac{2^{2-r}(3-r)}{r-1} \varepsilon + C(\varepsilon) \beta |r-2| + \varepsilon \right] \int_{A_{k,t}} |Dv|^r dx \\
+ C(\varepsilon) \int_{A_{k,t}} |F|^r dx + \frac{C(\varepsilon)}{(t-\tau)^r} \int_{A_{k,t}} |v|^r dx.
\]

Take \( \varepsilon \) sufficiently small, and then \( r \) sufficiently close to 2 such that \( \beta \frac{2^{2-r}(3-r)}{r-1} \varepsilon + C(\varepsilon) \beta |r-2| + \varepsilon < \alpha \), then Lemma 1.3 yields that for any \( R_0 \leq \rho < R \leq R_1 \),

\[
\int_{A_{k,\rho}} |Dv|^r dx \leq C \int_{A_{k,R}} |F|^r dx + \frac{C}{(R-\rho)^r} \int_{A_{k,R}} |v|^r dx.
\]

By Lemma 1.4 we have \( v \in L^{m^*}_{loc}(\Omega) \). This is equivalent to \( u \in L^{m^*(1-\theta)}_{loc}(\Omega) \). This ends the proof of Theorem 1.2.

References


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