Inequalities for the incomplete beta function

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Abstract

In this paper, we present some inequalities involving the incomplete beta function.

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1 Introduction

The beta function is defined by

\[ \beta(a, b) = \int_0^\infty \frac{t^{a-1}}{(1 + t)^{a+b}} dt, \]

where \(a, b > 0\).

The incomplete beta function is defined by

\[ \beta(a, b, x) = \int_x^\infty \frac{t^{a-1}}{(1 + t)^{a+b}} dt, \]

where \(a, b, x > 0\). We let \(\beta(a, b, 0) = \beta(a, b)\).

In 2010, Sulaiman [1] gave the inequalities as follows.

\[ \beta(a, b, x)\beta(a, b, y) \geq \beta(a, b, xy)\beta(a, b, 1) \]

where \(a, b > 0\) and \(x, y > 1\).

\[ \beta(a, b, x)\beta(a, b, y) \leq \beta(a, b, xy)\beta(a, b, 1) \]

where \(a, b > 0\) and \(0 < y < 1 < x\).
\[ \beta(a, b, x) \beta(a, b, y) \leq \beta(a, b, x + y) \beta(a, b, 0) \]  
(3)

where \( 0 < a < 1 \) and \( b, x, y > 0 \).

In this paper, we present the generalizations for the inequalities (1), (2) and (3).

2 Results

Theorem 2.1. Let \( a, b, c > 0 \) and \( x, y > c \). Then

\[ \beta(a, b, x) \beta(a, b, y) \geq \beta(a, b, \frac{xy}{c}) \beta(a, b, c). \]  
(4)

Proof. Let \( g(t) = \frac{t^{a-1}}{(1 + t)^{a+b}}, F(t) = \frac{\beta(a, b, t)}{\beta(a, b, c)} \) and \( G(t) = F(t)F(y) - F\left(\frac{ty}{c}\right) \) for all \( t > 0 \).

Then, for all \( t > 0 \),

\[ G'(t) = F'(t)F(y) - \frac{y}{c} F'\left(\frac{ty}{c}\right) \]
\[ = g(t)F(y) \left( \frac{yg\left(\frac{ty}{c}\right)}{cF(y)g(t)} - 1 \right) \]
\[ = g(t)F(y) \left( \frac{y^a}{c^aF(y)} \left( \frac{1 + t}{1 + \frac{ty}{c}} \right)^{a+b} - 1 \right). \]

We note that \( \left( \frac{1 + t}{1 + \frac{ty}{c}} \right)^{a+b} \) is decreasing in \( t > 0 \) since \( y > c \).

Let \( H(t) = \frac{y^a}{c^aF(y)} \left( \frac{1 + t}{1 + \frac{ty}{c}} \right)^{a+b} - 1 \) for all \( t > 0 \). Then \( H \) is decreasing.

We note that \( G(c) = F(c)F(y) - F(y) = 0 \) and \( \lim_{t \to \infty} G(t) = 0 \).

By Roll’s theorem, there is a point \( p \in (c, \infty) \) such that \( G'(p) = 0 \). Then \( H(p) = 0 \). Then \( H(t) > 0 \) for all \( t \in (c, p) \) and \( H(t) < 0 \) for all \( t \in (p, \infty) \). Then \( G'(t) > 0 \) for all \( t \in (c, p) \) and \( G'(t) < 0 \) for all \( t \in (p, \infty) \). This implies that \( G(x) \geq 0 \). Then \( F(x)F(y) \geq F\left(\frac{ty}{c}\right) \). Hence, we obtain the inequality (4).

We note on Theorem 2.1 that if \( c = 1 \) then we obtain the inequality (1).
Theorem 2.2. Let $a, b > 0$ and $0 < y < c < x$. Then

$$\beta(a, b, x)\beta(a, b, y) \leq \beta(a, b, \frac{xy}{c})\beta(a, b, c).$$ \hspace{1cm} (5)$$

Proof. Let $g(t) = \frac{t^{a-1}}{1 + t} \beta(a, b, t)$ and $G(t) = F(\frac{ty}{c}) - F(t)F(y)$ for all $t > 0$.

Then, for all $t > 0$,

$$G'(t) = \frac{y}{c} F'(\frac{ty}{c}) - F'(t)F(y)$$

$$= \frac{g(t)F(y)}{\beta(a, b, c)} \left(1 - \frac{y\frac{ty}{c}}{cF(y)g(t)}\right)$$

$$= \frac{g(t)F(y)}{\beta(a, b, c)} \left(1 - \frac{y^a}{c^aF(y)} \left(\frac{1 + t}{1 + \frac{ty}{c}}\right)^{a+b}\right).$$

We note that $\left(\frac{1 + t}{1 + \frac{ty}{c}}\right)^{a+b}$ is increasing in $t > 0$ since $y < c$.

Let $H(t) = 1 - \frac{y^a}{c^aF(y)} \left(\frac{1 + t}{1 + \frac{ty}{c}}\right)^{a+b}$ for all $t > 0$. Then $H$ is decreasing.

We note that $G(c) = F(y) - F(c)F(y) = 0$ and $\lim_{t \to \infty} G(t) = 0$.

By Roll’s theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \geq 0$. Then $F(\frac{x}{c}) \geq F(x)F(y)$. Hence, we obtain the inequality (5). \hfill \Box

We note on Theorem 2.2 that if $c = 1$ then we obtain the inequality (2).

Theorem 2.3. Let $0 < a < 1$, $b > 0$, $0 \leq c < y$ and $x > c$. Then

$$\beta(a, b, x)\beta(a, b, y) \leq \beta(a, b, x + y - c)\beta(a, b, c).$$ \hspace{1cm} (6)$$

Proof. For any $t > 0$, we let $g(t) = \frac{t^{a-1}}{1 + t} \beta(a, b, t)$ and $G(t) = F(t + y - c) - F(t)F(y)$. 

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Then, for all $t > 0$, 

$$G'(t) = F'(t + y - c) - F'(t)F(y)$$

$$= \frac{g(t)F(y)}{\beta(a, b, c)} \left( 1 - \frac{g(t + y - c)}{F(y)g(t)} \right)$$

$$= \frac{g(t)F(y)}{\beta(a, b, c)} \left( 1 - \frac{1}{F(y)} \left( 1 + \frac{y - c}{t} \right)^{a-1} \left( 1 + \frac{y - c}{1 + t} \right)^{-a-b} \right).$$

We note that $\left( 1 + \frac{y - c}{t} \right)^{a-1} \left( 1 + \frac{y - c}{1 + t} \right)^{-a-b}$ is increasing in $t > 0$ since $a < 1$ and $y > c$.

Let $H(t) = 1 - \frac{1}{F(y)} \left( 1 + \frac{y - c}{t} \right)^{a-1} \left( 1 + \frac{y - c}{1 + t} \right)^{-a-b}$ for all $t > 0$. Then $H$ is decreasing.

We note that $G(c) = F(y) - F(c)F(y) = 0$ and $\lim_{t \to \infty} G(t) = 0$.

By Roll’s theorem, there is a point $p \in (c, \infty)$ such that $G'(p) = 0$. Then $H(p) = 0$. Then $H(t) > 0$ for all $t \in (c, p)$ and $H(t) < 0$ for all $t \in (p, \infty)$. Then $G'(t) > 0$ for all $t \in (c, p)$ and $G'(t) < 0$ for all $t \in (p, \infty)$. This implies that $G(x) \geq 0$. Then $F(x + y - c) \geq F(x)F(y)$. Hence, we obtain the inequality (6).

We note on Theorem 2.3 that if $c = 0$ then we obtain the inequality (3).

References


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