Hom-pre-Lie algebras of three dimension

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Abstract

In this paper, we give some examples of Hom-pre-Lie algebras of three dimension. They can be obtained from pre-Lie algebras by twisting along any algebra endomorphism.

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Keywords: Lie algebra, Pre-Lie algebra, Hom-pre-Lie algebra

1 Preliminaries

Pre-Lie algebras were introduced in the studies of different geometry and Lie group, which have a close relationship with the Lie algebras. They have already been introduced by Cayley in 1896 as a kind of rooted tree algebras [5]. They affine manifolds and affine structures on Lie group [9,10]. They also play very important roles in the studies of certain in tegrable systems, classical and quantum Yang-Baxter equations [6,8], and so on. The Pre-Lie algebra is also called a left-symmetric algebra or Vinberg algebra. Hom-pre-Lie algebras were first introduced by Makblouf and Silvestor as a special case of G-Hom-associative algebras [4]. Pre-Lie algebra appears in many fields in mathematics and mathematical physics. So far, there have been many results in the study of Hom-pre-Lie algebra, but there are still some problems that have not been solved. In this paper, we shall give some examples of the 3-dimensional Hom-pre-Lie algebras.

Definition 1.1 [1] Let \( A \) be a vector space over a field \( F \) equipped with a bilinear product \( \mu : A \otimes A \to A \). \( A \) is called a pre-Lie-algebra if for any \( x, y, z \in A \),

\[(xy) z - x (yz) = (yx) z - y (xz) , \quad (1)\]
Definition 1.2 [2] Let $A$ be a vector space over a field $F$ equipped with a bilinear product $\mu : A^\otimes 2 \to A$. $A$ is called a pre-Lie-algebra if for any $x, y, z \in A$,

$$\alpha(xy) = \alpha(x) \alpha(y), \quad (2)$$

$$(xy)\alpha(z) - \alpha(x)(yz) = (yx)\alpha(z) - \alpha(y)(xz), \quad (3)$$

Definition 1.3 [7] Let $A$ be a vector space over a field $F$, $\mu : A^\otimes 2 \to A$ is bilinear map, and $\alpha : A \to A$ is a linear map. $(A, \mu, \alpha)$ is called Hom-Novikov algebra if the linear map satisfies equation (2) and (3), and the following condition for $x, y, z \in A$:

$$(xy)\alpha(z) = (xz)\alpha(y), \quad (4)$$

Definition 1.4 [3] Let $A$ be a vector space over a field $F$, $\mu : A^\otimes 2 \to A$ be bilinear map, and $\alpha : A \to A$ be a linear map. $(A, \mu, \alpha)$ is called G-Hom-associative algebra if any $x_i \in A (i = 1, 2, 3)$:

$$\sum_{\sigma \in G} (-1)^{\varepsilon(\sigma)} \left\{ (x_{\sigma(1)}x_{\sigma(2)})\alpha(x_{\sigma(3)}) - \alpha(x_{\sigma(1)})(x_{\sigma(2)}x_{\sigma(3)}) \right\}, \quad (5)$$

where $\varepsilon(\sigma)$ is the signature of $\sigma$.

1. A Hom-associative algebra is a G-Hom-associative algebra in which $G$ is the trivial subgroup $\{e\}$. The G-Hom-associativity now takes the form

$$(xy)\alpha(z) = \alpha(x)(yz), \quad (6)$$

which we call Hom-associativity.

2. A Hom-pre-Lie algebra is a G-Hom-associative algebra in which $G = \{e, (12)\}$. The $\{e, (1, 2)\}$ Hom-associativity axiom (5) is equivalent to

$$(xy)\alpha(x) - \alpha(x)(yz) = (yx)\alpha(z) - \alpha(y)(xz). \quad (7)$$

Theorem 1.5 [3] Let $A = (A, \mu)$ be a G-associative algebra and $\alpha : A \to A$ be a linear map such that $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. Then $(A, \mu_\alpha = \alpha \circ \mu, \alpha)$ is a G-Hom-associative algebra. Moreover, $\alpha$ is multiplicative with respect to $\mu_\alpha, i.e.,$

$$\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}. \quad (8)$$

Theorem 1.6 [3] Let $A = (A, \mu)$ be a not necessarily associative algebra and $\alpha : A \to A$ be a algebra morphism. Write $A_\alpha$ for the triple $(A_\alpha, \alpha_{\otimes 2} = \alpha \circ \mu, \alpha)$.

1. If $A$ is an associative algebra, then $A_\alpha$ is a Hom-associative algebra;
2. If $A$ is a Lie algebra, then $A_\alpha$ is a Hom-Lie algebra;
3. If $A$ is a pre-Lie algebra, then $A_\alpha$ is a Hom-pre-Lie algebra;
4. If $A$ is a Lie-admissible algebra, then $A_\alpha$ is a Hom-pre-Lie algebra.
2 Main results

Let $A$ be a 3-dimensional Lie algebra with a fixed linear basis $\{e_1, e_2, e_3\}$, and the characteristic matrix of $A$ be

$$M (\mu) = \begin{pmatrix} e_1 e_1 & e_1 e_2 & e_1 e_3 \\ e_2 e_1 & e_2 e_2 & e_2 e_3 \\ e_3 e_1 & e_3 e_2 & e_3 e_3 \end{pmatrix},$$  \hspace{1cm} (8)

where $e_i e_j = \sum_{k=1}^{3} d_{ij}^k e_k$. If $\alpha : A \to A$ is an algebra morphism and the $\mu_\alpha = \alpha \circ \mu$ is the associated Hom-pre-Lie algebra product, then its characteristic matrix is defined similarly as

$$M (\mu_\alpha) = \begin{pmatrix} \alpha (e_1 e_1) & \alpha (e_1 e_2) & \alpha (e_1 e_3) \\ \alpha (e_2 e_1) & \alpha (e_2 e_2) & \alpha (e_2 e_3) \\ \alpha (e_3 e_1) & \alpha (e_3 e_2) & \alpha (e_3 e_3) \end{pmatrix},$$  \hspace{1cm} (9)

We denote a linear map $\alpha : A \to A$ by its $3 \times 3$-matrix with respect to the basis $\{e_1, e_2, e_3\}$ and its matrix is

$$M (\alpha) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$  \hspace{1cm} (10)

where $\alpha (e_i) = \sum_{k=1}^{3} a_{ik} e_k, 1 \leq i \leq 3$.

The classification of pre-Lie algebras of dimension three given in [3] is divided five classes, $A$, $H$, $N$, $D$ and $E$. Together with $H$ and $E$ associated Hom-pre-Lie products (Theorem 2.2), the classifications of algebra morphisms on 3-dimensional pre-Lie algebras are listed in the table 1.

$$\alpha (e_1 e_2) = \alpha (e_1) \alpha (e_2) = (a_{11} e_1 + a_{21} e_2 + a_{31} e_3) (a_{11} e_1 + a_{21} e_2 + a_{31} e_3)$$  \hspace{1cm} (11)

And because $\alpha (e_1 e_1) = \alpha (e_1)$, then $a_{11}^2 e_1 + (a_{11} a_{21} + a_{21} a_{11}) e_2 + (a_{11} a_{21} + 2 a_{31} a_{11}) e_3 = a_{11} e_1 + a_{21} e_2 + a_{31} e_3$. We can get the equations

$$a_{11}^2 = a_{11}, 2 a_{11} a_{21} = a_{21}, a_{31} = a_{11} a_{21} + 2 a_{11} a_{31}.$$  

According to the $\alpha (e_1 e_2) = \alpha (e_2) + \alpha (e_3)$, we get the equations

$$a_{11} a_{12} = a_{12} + a_{13}, a_{11} a_{22} + a_{21} a_{12} = a_{22} + a_{23}, a_{12} a_{22} + a_{11} a_{32} + a_{31} a_{12} = a_{32} + a_{33}.$$
Table 1: The classifications of algebra morphisms on 3-dimensional pre-Lie algebras

<table>
<thead>
<tr>
<th>Pre-Lie algebra $M(\mu)$</th>
<th>Algebra morphism $M(\mu)$</th>
<th>Hom-pre-Lie algebra $M(\mu)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(H - 1) = \begin{pmatrix} e_1 &amp; e_2 &amp; e_3 \ e_2 &amp; 0 &amp; 0 \ e_3 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; b_2 &amp; 0 \ 0 &amp; b_3 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} e_1 &amp; b_2 e_2 + b_3 e_3 &amp; 0 \ b_2 e_2 &amp; b_3 e_3 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(H - 2) = \begin{pmatrix} e_1 &amp; e_2 + e_3 &amp; e_3 \ e_2 &amp; 0 &amp; 0 \ e_3 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; b_3 &amp; 0 \ 0 &amp; b_3 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} e_1 &amp; b_3 e_3 &amp; 0 \ b_3 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(H - 3) = \begin{pmatrix} e_1 &amp; e_3 &amp; 0 \ e_2 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; b_3 &amp; 1 \ 0 &amp; b_3 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} e_1 &amp; e_3 &amp; 0 \ e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(H - 4) = \begin{pmatrix} e_1 &amp; e_3 &amp; 0 \ e_2 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; b_3 &amp; b_2 \ 0 &amp; b_2 &amp; b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} e_1 &amp; b_2 e_3 &amp; 0 \ b_2 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(H - 5) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ -e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} a_1 &amp; b_1 &amp; 0 \ a_2 &amp; b_2 &amp; 0 \ a_3 &amp; b_3 &amp; a_1 b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ -a_1 b_2 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(H - 6) = \begin{pmatrix} 0 &amp; e_3 &amp; 0 \ 0 &amp; e_3 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} b_3 &amp; b_3 &amp; 0 \ 0 &amp; b_3 &amp; 0 \ -b_3 b_2 &amp; b_3 b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ -b_3 b_3 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(H - 7) = \begin{pmatrix} e_3 &amp; e_3 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$, $\lambda \neq 0$</td>
<td>$\begin{pmatrix} a_1 &amp; b_1 &amp; 0 \ a_2 &amp; b_2 &amp; 0 \ a_3 &amp; b_3 &amp; 2 a_1 b_2 e_3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 2 a_1 b_2 e_3 &amp; 2 a_1 b_2 e_3 &amp; 0 \ 2 a_1 b_2 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(H - 8) = \begin{pmatrix} 0 &amp; e_3 &amp; 0 \ 0 &amp; e_3 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} b_3 &amp; b_3 &amp; 0 \ 0 &amp; b_3 &amp; 0 \ b_3 b_2 &amp; b_3 b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ b_3 b_3 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(H - 9) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; e_1 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} b_3 &amp; b_3 &amp; 0 \ 0 &amp; b_3 &amp; 0 \ b_1 b_2 &amp; b_3 b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ b_3 b_3 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(H - 10) = \begin{pmatrix} 0 &amp; e_3 &amp; 0 \ 0 &amp; e_3 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$, $(\lambda \neq 0, 1)$</td>
<td>$\begin{pmatrix} b_3 &amp; b_3 &amp; 0 \ 0 &amp; b_3 &amp; 0 \ b_3 b_2 &amp; b_3 b_2 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ b_3 b_3 e_3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(E - 1) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$, $\lambda \neq 0$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(E - 2) = \begin{pmatrix} 0 &amp; e_1 + e_2 &amp; e_3 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} b_1 c_1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} b_1 e_1 + e_2 + (b_1 + c_1) e_3 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(E - 3) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; e_1 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} a_1 b_1 &amp; 0 \ a_1 b_1 &amp; 0 \ a_1 b_1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$(E - 4) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; e_1 &amp; (\lambda + 1) e_1 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$, $(\lambda \neq 0, -1)$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
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<td>$(E - 5) = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; e_3 &amp; 0 \ 0 &amp; e_1 &amp; e_2 + e_3 \end{pmatrix}$</td>
<td>$\begin{pmatrix} a_1 b_1 &amp; 0 \ a_1 b_1 &amp; 0 \ a_1 b_1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 0 &amp; 0 &amp; 0 \ a_1 b_1 (\lambda + 1) e_1 + b_1 e_1 + a_1 c_2 + a_1 c_2 \ a_1 b_1 (\lambda + 1) e_1 + b_1 e_1 + a_1 c_2 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(E - 6) = \begin{pmatrix} 0 &amp; e_3 &amp; 0 \ 0 &amp; e_1 - e_3 - e_2 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
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</tbody>
</table>
According to the $\alpha(e_1 e_3) = \alpha(e_3)$, we get the equations

$$a_{11} a_{13} = a_{13}, a_{11} a_{23} + a_{21} a_{13} = a_{23}, a_{11} a_{23} + a_{11} a_{33} + a_{31} a_{13} = a_{33}$$

According to the $\alpha(e_2 e_1) = \alpha(e_2)$, we get the equations

$$a_{12} a_{11} = a_{12}, a_{12} a_{21} + a_{22} a_{11} = a_{22}, a_{12} a_{21} + a_{12} a_{31} + a_{32} a_{11} = a_{32}.$$

According to the $\alpha(e_1 e_2) = 0$, we get the equations

$$a_{11}^2 = 0, 2a_{12} a_{22} = 0, a_{12} a_{22} + 2a_{12} a_{32} = 0.$$

According to the $\alpha(e_2 e_3) = 0$, we get the equations

$$a_{12} a_{13} = 0, a_{12} a_{23} + a_{22} a_{13} = 0, a_{12} a_{23} + a_{12} a_{33} + a_{32} a_{13} = 0.$$

According to the $\alpha(e_3 e_1) = \alpha(e_3)$, we get the equations

$$a_{13} a_{11} = a_{13}, a_{13} a_{21} + a_{23} a_{11} = a_{23}, a_{33} = a_{13} a_{21} + a_{13} a_{31} + a_{33} a_{11}.$$

According to the $\alpha(e_3 e_2) = 0$, we get the equations

$$a_{13} a_{12} = 0, a_{13} a_{22} + a_{23} a_{12} = 0, a_{13} a_{22} + a_{13} a_{32} + a_{33} a_{12} = 0.$$

According to the $\alpha(e_3 e_3) = 0$, we get the equations

$$a_{13}^2 = 0, 2a_{13} a_{23} = 0, a_{13} a_{23} + 2a_{13} a_{33} = 0.$$

So we obtain the matrixes of $\alpha$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & b_2 & 0 \end{pmatrix}.$$

As if $e_i e_j = \sum_{k=1}^{3} d_{ij}^{k} e_k$, and the corresponding characteristic matrixes are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} e_1 & b_2 e_2 + b_3 e_3 & 0 \\ b_2 e_2 + b_3 e_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

References


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