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Higher Integrability for Solutions to Nonhomogeneous Elliptic Systems in Divergence Form

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Abstract

In this paper we obtain a higher integrability result for weak solutions to nonhomogeneous elliptic systems in divergence form under some suitable assumptions.

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1 Introduction and Statement of Result

Let $\Omega \subset \mathrm{R}^n$, $n \geq 3$, be a bounded domain and $N \geq 2$ be an integer. For $i, j = 1, 2, \cdots, n$ and $\alpha, \beta = 1, 2, \cdots, N$, we let the Carathéodory functions $a_{ij}^{\alpha\beta}(x, y) : \Omega \times \mathrm{R}^N \to \mathrm{R}$ to be bounded, i.e., there exists a constant c_1 such that

$$\left| a_{ij}^{\alpha\beta}(x, y) \right| \leq c_1, \tag{1.1}$$

for almost every $x \in \Omega$, for every $y \in \mathrm{R}^N$, for all $i, j = 1, 2, \cdots, n$, and for all $\alpha, \beta = 1, 2, \cdots, N$.

We consider weak solutions $u = (u^1, \cdots, u^N) \in W^{1,2}(\Omega, \mathrm{R}^N)$ to nonhomogeneous elliptic systems in divergence form

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = -\sum_{i=1}^{n} D_i f_i^\alpha(x, u(x)), \ \ x \in \Omega, \ \ \alpha = 1, \cdots, N. \tag{1.2}$$

Definition 1.1 *A function $u \in W^{1,2}(\Omega, R^N)$ is called a weak solution to (1.2), if*

$$\int_\Omega \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) D_i v^\alpha(x) dx = \int_\Omega \sum_{i=1}^{n} \sum_{\alpha=1}^{N} f_i^\alpha(x, u(x)) D_i v^\alpha(x) dx \tag{1.3}$$

holds true for any $v \in W_0^{1,2}(\Omega, R^N)$.

In [1] the authors consider homogeneous elliptic system

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\alpha\beta}(x, u(x)) D_j u^\beta(x) \right) = 0, \quad x \in \Omega, \quad \alpha = 1, \cdots, N,$$

and assume, besides the ellipticity condition of diagonal coefficients, that off-diagonal coefficients $a_{ij}^{\gamma\beta}(x, y)$, $i, j = 1, 2, \cdots, n$, $\gamma, \beta = 1, 2, \cdots, N$, $\gamma \neq \beta$, are small when the corresponding component y^γ is large, i.e., there exists $q, c_2 > 0$ such that

$$0 < \theta^\gamma \leq |u^\gamma| \Rightarrow |a_{ij}^{\gamma\beta}(x, y)| \leq \frac{c_2}{|y^\gamma|^q} \quad \text{for } \beta \neq \gamma. \tag{1.4}$$

The authors obtained an estimate for the measure of the superlevel set: for every $s > 0$,

$$|\{|u^\gamma| > s\}| \leq \frac{c_3}{s^{2^*(1+q)}},$$

where c_3 is a constant. That is, u^γ is in the weak Lebesgue space with exponent $2^*(1+q)$,

$$u^\gamma \in L_{weak}^{2^*(1+q)}(\Omega).$$

See Theorem 2.2 in [1]. For other related works, see [2-7]. In the present paper, we deal with integrability property for weak solutions u to the nonhomogeneous elliptic system (1.1). We assume ellipticity of diagonal coefficients $a_{ij}^{\gamma\gamma}(x, y)$ for large values of the corresponding component of y: for $\gamma = 1, 2, \cdots, N$, there exists $\theta^\gamma > 0$ such that

$$\theta^\gamma \leq |y^\gamma| \Rightarrow \nu |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x, y) \xi_i \xi_j \tag{1.5}$$

for almost every $x \in \Omega$ and for any $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathrm{R}^n$, where $\nu > 0$ is a constant. For off-diagonal coefficients $a_{ij}^{\gamma\beta}(x, y)$, $i, j = 1, 2, \cdots, n$, $\gamma, \beta =$

$1, 2, \cdots, N$, $\gamma \neq \beta$, we assume (1.4). For the terms $f_i^\gamma(x, u(x))$, $\gamma = 1, 2, \cdots, N$, we assume that they are small when the corresponding component y^γ is large:

$$0 < \theta^\gamma \leq |u^\gamma| \Rightarrow |f_i^\gamma(x, y)| \leq \frac{c_2}{|y^\gamma|^q}, \quad \gamma = 1, 2, \cdots, N. \tag{1.6}$$

The main result of this paper is

Theorem 1.2 *Suppose that for every* $\gamma = 1, 2, \cdots, N$,

$$-\infty < \inf_{\partial\Omega} u^\gamma \quad \text{and} \quad \sup_{\partial\Omega} u^\gamma < +\infty. \tag{1.7}$$

Under the previous assumptions (1.1), (1.4), (1.5) and (1.6), let $u = (u^1, u^2, \cdots, u^N)$ *be a weak solution of the system (1.2), then* u *attains higher integrability*

$$u \in L^{2^*(1+q)}_{weak}(\Omega, R^N).$$

2 Proof of Theorem 1.1.

For every $\gamma \in \{1, 2, \cdots, N\}$, for every $L \in (0, +\infty)$, we define

$$g_1^\gamma(L) = \max_{i,j} \max_{\beta \neq \gamma} \sup_{|y^\gamma| > L} \sup_x \left| a_{ij}^{\gamma\beta}(x, y) \right| \tag{2.1}$$

and

$$g_2^\gamma(L) = \max_i \sup_{|y^\gamma| > L} \sup_x |f_i^\gamma(x, y)|. \tag{2.2}$$

By the conditions (1.1), (1.4) and (1.6) we have

$$0 < g_1^\gamma(L), g_2^\gamma(L) \leq \min\left\{c_1, \frac{c_2}{L^q}\right\}. \tag{2.3}$$

Our nearest goal is to prove that for every $\gamma = 1, 2, \cdots, N$, the estimate

$$|\{x \in \Omega : |u^\gamma(x)\}| > 2L| \leq \left(c_4 \frac{g_1^\gamma(L)}{L} + c_5 \frac{g_2^\gamma(L)}{L}\right)^{2^*}, \tag{2.4}$$

holds true for $L \geq \max\{\theta^\gamma, \sup_{\partial\Omega} u^\gamma, -\inf_{\partial\Omega} u^\gamma\}$, where

$$c_4 = \frac{2(n-1)}{n-2} \frac{n^2(N-1)\|Du\|_{L^2(\Omega)}}{\nu}, \quad c_5 = \frac{2(n-1)}{n-2} \frac{n|\Omega|^{\frac{1}{2}}}{\nu},$$

and u^γ is the γ-th component of $u = (u^1, \cdots, u^N)$. Since $L \geq \sup_{\partial\Omega} u^\gamma$, we have $(u^\gamma - L)^+ \in W_0^{1,2}(\Omega)$. We define $v = (v^1, v^2, \cdots, v^N)$ as follows

$$\begin{cases} v^\alpha = 0, & \text{if } \alpha \neq \gamma, \\ v^\gamma = (u^\gamma - L)^+, & \text{otherwise.} \end{cases} \tag{2.5}$$

This implies

$$\begin{cases} Dv^{\alpha} = 0, & \text{if } \alpha \neq \gamma, \\ Dv^{\gamma} = 1_{\{u^{\gamma}>L\}} Du^{\gamma}, & \text{otherwise}, \end{cases} \tag{2.6}$$

where 1_E is the characteristic function of the set E. We insert such a test function v into (1.3),

$$\begin{aligned} & \int_{\{u^{\gamma}>L\}} \sum_{i=1}^{n} f_i^{\gamma}(x,u(x)) D_i u^{\gamma}(x) dx \\ = & \int_{\Omega} \sum_{i=1}^{n} \sum_{\alpha=1}^{N} f_i^{\alpha}(x,u(x)) D_i v^{\alpha}(x) dx \\ = & \int_{\Omega} \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} a_{ij}^{\alpha\beta}(x,u(x)) D_j u^{\beta}(x) D_i v^{\alpha}(x) dx \\ = & \int_{\Omega} \sum_{i,j=1}^{n} \sum_{\beta=1}^{N} a_{ij}^{\gamma\beta}(x,u(x)) D_j u^{\beta}(x) 1_{\{u^{\gamma}>L\}}(x) D_i u^{\gamma}(x) dx \\ = & \int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u(x)) D_j u^{\gamma}(x) D_i u^{\gamma}(x) dx \\ & + \int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u(x)) D_j u^{\beta}(x) D_i u^{\gamma}(x) dx. \end{aligned}$$

Then

$$\begin{aligned} & \int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u(x)) D_j u^{\gamma}(x) D_i u^{\gamma}(x) dx \\ = & -\int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u(x)) D_j u^{\beta}(x) D_i u^{\gamma}(x) dx \\ & + \int_{\{u^{\gamma}>L\}} \sum_{i=1}^{n} f_i^{\gamma}(x,u(x)) D_i u^{\gamma}(x) dx. \end{aligned} \tag{2.7}$$

Since $L \geq \theta^{\gamma}$, we can use ellipticity (1.5) and we get

$$\int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} a_{ij}^{\gamma\gamma}(x,u(x)) D_j u^{\gamma}(x) D_i u^{\gamma}(x) dx \geq \nu \int_{\{u^{\gamma}>L\}} |Du^{\gamma}|^2 dx. \tag{2.8}$$

We keep in mind the definitions (2.1) for $g_1^{\gamma}(L)$ and (2.2) for $g_2^{\gamma}(L)$ and we derive

$$\begin{aligned} & -\int_{\{u^{\gamma}>L\}} \sum_{i,j=1}^{n} \sum_{\beta\neq\gamma} a_{ij}^{\gamma\beta}(x,u(x)) D_j u^{\beta}(x) D_i u^{\gamma}(x) dx \\ \leq & \; n^2(N-1) g_1^{\gamma}(L) \int_{\{u^{\gamma}>L\}} |Du||Du^{\gamma}| dx \end{aligned} \tag{2.9}$$

and

$$\int_{\{u^{\gamma}>L\}} \sum_{i=1}^{n} f_i^{\gamma}(x,u(x)) D_i u^{\gamma}(x) dx \leq n g_2^{\gamma}(L) \int_{\{u^{\gamma}>L\}} |Du^{\gamma}| dx. \tag{2.10}$$

Equality (2.7) and estimates (2.8), (2.9), (2.10) merge into

$$\nu \int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \le n^2(N-1) g_1^\gamma(L) \int_{\{u^\gamma > L\}} |Du||Du^\gamma| dx + n g_2^\gamma(L) \int_{\{u^\gamma > L\}} |Du^\gamma| dx. \tag{2.11}$$

We use Hölder's inequality on the right hand side of (2.11) in order to get

$$\begin{aligned} & \nu \int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \\ \le\; & n^2(N-1) g_1^\gamma(L) \left(\int_{\{u^\gamma > L\}} |Du|^2 dx \right)^{\frac{1}{2}} \left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \right)^{\frac{1}{2}} \\ & + n g_2^\gamma(L) |\{u^\gamma > L\}|^{\frac{1}{2}} \left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

We divide both sides by $\left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \right)^{\frac{1}{2}}$ and using the fact $|\{u^\gamma > L\}| \le |\Omega|$ we get

$$\left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \right)^{\frac{1}{2}} \le \frac{n^2(N-1) g_1^\gamma(L)}{\nu} \left(\int_{\{u^\gamma > L\}} |Du|^2 dx \right)^{\frac{1}{2}} + \frac{n g_2^\gamma(L) |\Omega|^{\frac{1}{2}}}{\nu}. \tag{2.12}$$

We keep in mind that $v^\gamma = (u^\gamma - L)^+ \in W_0^{1,2}(\Omega)$ and $n \ge 3$, thus Sobolev inequality and (2.12) allow us to write

$$\begin{aligned} & \int_{\{u^\gamma > L\}} (u^\gamma - L)^{2^*} dx \\ =\; & \|v^\gamma\|_{L^{2^*}(\Omega)}^{2^*} \le \left(\frac{2(n-1)}{n-2} \|Dv^\gamma\|_{L^2(\Omega)} \right)^{2^*} \\ =\; & \left(\frac{2(n-1)}{n-2} \left(\int_{\{u^\gamma > L\}} |Du^\gamma|^2 dx \right)^{\frac{1}{2}} \right)^{2^*} \\ \le\; & \left[\frac{2(n-1)}{n-2} \left(\frac{n^2(N-1) g_1^\gamma(L) \|Du\|_{L^2(\Omega)}}{\nu} + \frac{n g_2^\gamma(L) |\Omega|^{\frac{1}{2}}}{\nu} \right) \right]^{2^*} \\ =\; & \left(c_4 g_1^\gamma(L) + c_5 g_2^\gamma(L) \right)^{2^*}. \end{aligned} \tag{2.13}$$

Since $L > 0$, it turns out that $\{u^\gamma > 2L\} \subset \{u^\gamma > L\}$, thus

$$\begin{aligned} L^{2^*} |\{u^\gamma > 2L\}| &= \int_{\{u^\gamma > 2L\}} (2L - L)^{2^*} dx \\ &\le \int_{\{u^\gamma > 2L\}} (u^\gamma - L)^{2^*} dx \le \int_{\{u^\gamma > L\}} (u^\gamma - L)^{2^*} dx. \end{aligned} \tag{2.14}$$

Inequality (2.13) and (2.14) merge into

$$|\{|u^\gamma(x)| > 2L\}| \le \left(c_4 \frac{g_1^\gamma(L)}{L} + c_5 \frac{g_2^\gamma(L)}{L} \right)^{2^*}. \tag{2.15}$$

This estimate holds true for every $L \geq \max\{\theta^\gamma; \sup_{\partial\Omega} u^\gamma\} > 0$. Since $-\inf_{\partial\Omega} u^\gamma = \sup_{\partial\Omega}(-u^\gamma)$, if $L \geq \max\{\theta^\gamma; -\inf_{\partial\Omega} u^\gamma\} > 0$, then we can apply the previous inequality (2.15) to $-u$. This proves (2.4).

(2.4) together with (2.3) and the fact $|\Omega| < \infty$ gives us

$$|\{|u^\gamma| > 2L\}| \leq \frac{c_6}{L^{2^*(1+q)}},$$

where c_6 is a constant. This ends the proof of Theorem 1.1.

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