Hardy Inequality for $L^{θ,∞}$ Space

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Abstract

Hardy inequality for $L^{θ,∞}(I)$ space is proved. As a generalization, boundedness for Hardy-Littlewood maximal operator in $L^{θ,∞}(I)$ is derived.

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Let $I = (0, 1), θ ≥ 0$. The grand $L^∞$ space, $L^{θ,∞}(I)$, was introduced in [1] by

$$L^{θ,∞}(I) = \left\{ f(x) ∈ \bigcap_{1 ≤ p < ∞} L^p(I) : \sup_{1 ≤ p < ∞} \frac{1}{p} \left( \frac{1}{|J|} \int_J f^p dx \right)^{\frac{1}{p}} < ∞ \right\},$$

where $\frac{1}{|J|} \int_J f$ stands for integral mean over $J$, and $|J|$ denote the Lebesgue measure of $J$. It is known from [1] that $L^{0,∞}(I) = L^∞(I), L^{θ,∞}(I)$ is a generalization of the classical exponential class, and the following embedding holds

$L^∞(I) \subset L^{θ,∞}(I) \subset L^p(I)$
for any $\theta \geq 0$ and $1 < p < \infty$. For $f \in L^{(p,\infty)}(I)$, define

$$\|f\|_{L^{(p,\infty)}(I)} = \sup_{1 \leq p < \infty} \left\{ \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}} \right\}. \tag{1}$$

It is known that $\left( L^{(p,\infty)}(I), \| \cdot \|_{L^{(p,\infty)}(I)} \right)$ is a Banach space.

The classical Hardy inequality states that

Theorem 1. Let $p > 1$ and $f$ be a measurable, nonnegative function in $I$. Then

$$\left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left( \int_0^1 f^p \, dx \right)^{\frac{1}{p}}. \tag{2}$$

In other words, (2) is equivalent to

$$\left\| \int_0^x f \, dt \right\|_{L^p(I)} \leq \frac{p}{p-1} \|f\|_{L^p(I)}. \tag{2}'$$

The main result of this paper is the Hardy inequality in the $L^{(p,\infty)}(I)$ space.

Theorem 2. Let $0 < \theta < \infty$. Then

$$\left\| \int_0^x f \, dt \right\|_{L^{(p,\infty)}(I)} \leq \frac{(1 + \theta)^{1+\theta}}{\theta^\theta} \|f\|_{L^{(p,\infty)}(I)}. \tag{3}$$

Proof. For any $p_0 \in (1, \infty)$, one has

$$\left\| \int_0^x f \, dt \right\|_{L^{(p,\infty)}(I)} = \max \left\{ \sup_{1 \leq p < p_0} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}}, \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}} \right\}$$

$$\leq \max \left\{ \sup_{1 \leq p < p_0} \frac{p_0}{(pp_0)^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^{p_0} \, dx \right)^{\frac{1}{p_0}}, \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}} \right\}$$

$$\leq \max \left\{ \sup_{1 \leq p < p_0} \frac{p_0}{p^\theta} \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}}, \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}} \right\}$$

$$= \frac{p_0^\theta}{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 \left( \int_0^x f \, dt \right)^p \, dx \right)^{\frac{1}{p}} \leq \frac{p_0^\theta}{p_0 \leq p < \infty} \frac{1}{p_0^\theta(p-1)} \left( \int_0^1 f^p \, dx \right)^{\frac{1}{p}} \leq \frac{p_0^{1+\theta}}{p_0 - 1} \sup_{1 \leq p < \infty} \frac{1}{p^\theta} \left( \int_0^1 f^p \, dx \right)^{\frac{1}{p}},$$

where we have used Theorem 1. Since $\frac{p_0^{1+\theta}}{p_0 - 1}$ takes its minimum value $\frac{(1+\theta)^{1+\theta}}{\theta^\theta}$ at $p_0 = \frac{1+\theta}{p_0}$, then we set $p_0 = \frac{1+\theta}{p_0}$ getting the inequality (3).
As a corollary of Theorem 2, we derive the Hardy-Littlewood inequality for $L^{\theta,\infty}(I)$ space. The classical Hardy-Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{I \ni x} \int_I |f(t)| \, dt, \quad x \in (0, 1),$$

where the supremum extends over all non-degenerate intervals, contained in $I$, containing $x$.

For $f$ a measurable, nonnegative function in $I$, the decreasing rearrangement of $f$ is defined by

$$f^*(t) = \sup_{|E|=t} \inf_E f, \quad t \in (0, 1),$$

where the supremum extends over all measurable set $E \subset I$. An important relation between rearrangements and the maximal operator is given by the following Herz’s Theorem (see [2]), which establishes the equivalence of the function $(Mf)^*$ and the averaged rearrangement of $f$ defined by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t \in (0, 1).$$

**Theorem 3.** There are absolute constants $c$ and $c'$ such that the following inequalities hold for all $f \in L^1(I)$

$$c(Mf)^*(t) \leq f^{**}(t) \leq c'(Mf)^*(t), \quad t \in I.$$

**Corollary 1.** Let $0 < \theta < \infty$. Then

$$\|Mf\|_{L^{\theta,\infty}(0,1)} \leq C \|f\|_{L^{\theta,\infty}(0,1)}.$$

**Proof.** Since

$$\|f\|_p = \|f^*\|_p,$$

then from Theorem 2 and from Theorem 3 applied to $f^*$ we get

$$\|Mf\|_{L^{\theta,\infty}(I)} = \|(Mf)^*\|_{L^{\theta,\infty}(I)} \leq C \|f^{**}\|_{L^{\theta,\infty}(I)} \leq C \|f^*\|_{L^{\theta,\infty}(I)} \leq C \|f\|_{L^{\theta,\infty}(I)}$$

from which the assertion of Corollary 1 follows.

From the proof of Theorem 1 we know that a useful property of the norm (1) is in fact the supremum over $[1, \infty)$ in the norm of $L^{\theta,\infty}(I)$ can be computed also in any subinterval $(p_0, \infty), p_0 > 1$. The result is an equivalent expression of the norm (i.e., each expression can be majorized by the other, multiplied by a constant not depending on $f$).

**Theorem 4.** Let $1 < p_0 < \infty$. Then

$$\sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}} \leq \|f\|_{L^{\theta,\infty}(I)} \leq p_0^\theta \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}}.$$
Proof. The left wing inequality is trivial, therefore we need to prove only the right wing one. Since
\[
\sup_{1 \leq p < p_0} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}} \leq \sup_{1 \leq p < p_0} \frac{1}{p^\theta} \left( \int_I f^{p_0} \, dx \right)^{\frac{1}{p_0}} \\
\leq \sup_{1 \leq p < p_0} \frac{p_0^\theta}{p^\theta} \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}} = p_0^\theta \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}},
\]
then
\[
\|f\|_{L^{\infty}(I)} = \max \left\{ \sup_{1 \leq p < p_0} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}}, \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}} \right\} \\
\leq p_0^\theta \sup_{p_0 \leq p < \infty} \frac{1}{p^\theta} \left( \int_I f^p \, dx \right)^{\frac{1}{p}},
\]
as desired.

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References


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