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Hardy Inequality for $L^{\theta,\infty)}$ Space

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Abstract

Hardy inequality for $L^{\theta,\infty)}(I)$ space is proved. As a generalization, boundedness for Hardy-Littlewood maximal operator in $L^{\theta,\infty)}(I)$ is derived.

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Let $I = (0, 1)$, $\theta \geq 0$. The grand L^∞ space, $L^{\theta,\infty)}(I)$, was introduced in [1] by

$$L^{\theta,\infty)}(I) = \left\{ f(x) \in \bigcap_{1 \leq p < \infty} L^p(I) : \sup_{1 \leq p < \infty} \frac{1}{p^\theta} \left(⨍_I f^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

where $⨍_J = \frac{1}{|J|} \int_J$ stands for integral mean over J, and $|J|$ denote the Lebesgue measure of J. It is known from [1] that $L^{0,\infty)}(I) = L^\infty(I)$, $L^{\theta,\infty)}(I)$ is a generalization of the classical exponential class, and the following embedding holds

$$L^\infty(I) \subset L^{\theta,\infty)}(I) \subset L^p(I)$$

for any $\theta \geq 0$ and $1 < p < \infty$. For $f \in L^{\theta,\infty)}(I)$, define

$$\|f\|_{L^{\theta,\infty)}(I)} = \sup_{1\leq p<\infty} \frac{1}{p^\theta}\left(⨍_I f^p dx\right)^{\frac{1}{p}}. \tag{1}$$

It is known that $\left(L^{\theta,\infty)}(I), \|\cdot\|_{L^{\theta,\infty)}(I)}\right)$ is a Banach space.

The classical Hardy inequality states that

Theorem 1. *Let $p > 1$ and f be a measurable, nonnegative function in I. Then*

$$\left(\int_0^1 \left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^1 f^p dx\right)^{\frac{1}{p}}. \tag{2}$$

In other words, (2) is equivalent to

$$\left\|⨍_0^x f dt\right\|_{L^p(I)} \leq \frac{p}{p-1}\|f\|_{L^p(I)}. \tag{2$'$}$$

The main result of this paper is the Hardy inequality in the $L^{\theta,\infty)}(I)$ space.

Theorem 2. *Let $0 < \theta < \infty$. Then*

$$\left\|⨍_0^x f dt\right\|_{L^{\theta,\infty)}(I)} \leq \frac{(1+\theta)^{1+\theta}}{\theta^\theta}\|f\|_{L^{\theta,\infty)}(I)}. \tag{3}$$

Proof. For any $p_0 \in (1,\infty)$, one has

$$\begin{aligned}
&\left\|⨍_0^x f dt\right\|_{L^{\theta,\infty)}(I)} \\
= \;& \max\left\{\sup_{1\leq p<p_0}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}}, \sup_{p_0\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}}\right\} \\
\leq \;& \max\left\{\sup_{1\leq p<p_0}\frac{p_0^\theta}{(pp_0)^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^{p_0} dx\right)^{\frac{1}{p_0}}, \sup_{p_0\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}}\right\} \\
\leq \;& \max\left\{\sup_{1\leq p<p_0}\frac{p_0^\theta}{p^\theta}\sup_{p_0\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}}, \sup_{p_0\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}}\right\} \\
= \;& p_0^\theta \sup_{p_0\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1\left(⨍_0^x f dt\right)^p dx\right)^{\frac{1}{p}} \\
\leq \;& p_0^\theta \sup_{p_0\leq p<\infty}\frac{p}{p^\theta(p-1)}\left(\int_0^1 f^p dx\right)^{\frac{1}{p}} \\
\leq \;& \frac{p_0^{1+\theta}}{p_0-1}\cdot\sup_{1\leq p<\infty}\frac{1}{p^\theta}\left(\int_0^1 f^p dx\right)^{\frac{1}{p}},
\end{aligned}$$

where we have used Theorem 1. Since $\frac{p_0^{1+\theta}}{p_0-1}$ takes its minimum value $\frac{(1+\theta)^{1+\theta}}{\theta^\theta}$ at $p_0 = \frac{1+\theta}{\theta}$, then we set $p_0 = \frac{1+\theta}{\theta}$ getting the inequality (3).

As a corollary of Theorem 2, we derive the Hardy-Littlewood inequality for $L^{\theta,\infty)}(I)$ space. The classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{I\supset J\ni x} \fint_J |f(t)|dt, \quad x \in (0,1),$$

where the supremum extends over all non-degenerate intervals, contained in I, containing x.

For f a measurable, nonnegative function in I, the decreasing rearrangement of f is defined by

$$f^*(t) = \sup_{|E|=t} \inf_E f, \quad t \in I,$$

where the supremum extends over all measurable set $E \subset I$. An important relation between rearrangements and the maximal operator is given by the following Herz's Theorem (see [2]), which establishes the equivalence of the function $(Mf)^*$ and the averaged rearrangement of f defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t \in (0,1).$$

Theorem 3. *There are absolute constants c and c' such that the following inequalities hold for all $f \in L^1(I)$*

$$c(Mf)^*(t) \le f^{**}(t) \le c'(Mf)^*(t), \quad t \in I.$$

Corollary 1. *Let $0 < \theta < \infty$. Then*

$$\|Mf\|_{L^{\theta,\infty)}(0,1)} \le C\|f\|_{L^{\theta,\infty)}(0,1)}.$$

Proof. Since

$$\|f\|_p = \|f^*\|_p,$$

then from Theorem 2 and from Theorem 3 applied to f^* we get

$$\|Mf\|_{L^{\theta,\infty)}(I)} = \|(Mf)^*\|_{L^{\theta,\infty)}(I)} \le C\|f^{**}\|_{L^{\theta,\infty)}(I)} \le C\|f^*\|_{L^{\theta,\infty)}(I)} \le C\|f\|_{L^{\theta,\infty)}(I)}$$

from which the assertion of Corollary 1 follows.

From the proof of Theorem 1 we know that a useful property of the norm (1) is in fact the supremum over $[1,\infty)$ in the norm of $L^{\theta,\infty}(I)$ can be computed also in any subinterval (p_0,∞), $p_0 > 1$. The result is an equivalent expression of the norm (i.e., each expression can be majorized by the other, multiplied by a constant not depending on f).

Theorem 4. *Let $1 < p_0 < \infty$. Then*

$$\sup_{p_0\le p<\infty} \frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}} \le \|f\|_{L^{\theta,\infty)}(I)} \le p_0^\theta \sup_{p_0\le p<\infty} \frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}}.$$

Proof. The left wing inequality is trivial, therefore we need to prove only the right wing one. Since

$$\begin{aligned}
&\sup_{1\le p<p_0}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}} \le \sup_{1\le p<p_0}\frac{1}{p^\theta}\left(\int_I f^{p_0} dx\right)^{\frac{1}{p_0}}\\
\le &\sup_{1\le p<p_0}\frac{p_0^\theta}{p^\theta}\cdot\sup_{p_0\le p<\infty}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}} = p_0^\theta\sup_{p_0\le p<\infty}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}},
\end{aligned}$$

then

$$\begin{aligned}
\|f\|_{L^{\theta,\infty)}(I)} &= \max\left\{\sup_{1\le p<p_0}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}}, \sup_{p_0\le p<\infty}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}}\right\}\\
&\le p_0^\theta\sup_{p_0\le p<\infty}\frac{1}{p^\theta}\left(\int_I f^p dx\right)^{\frac{1}{p}},
\end{aligned}$$

as desired.

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