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Functions of Waterman-Shiba Bounded Variation with Variable Exponent

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Abstract

In this paper we introduce the notion of functions of bounded variation in the Waterman-Shiba's sense with variable exponent on a real interval $[a, b]$ and we study some of its basic properties. We show that the set of all such functions generalize some known Banach spaces. Finally we exhibit a rich subclass of composite functions that belong to the introduced class.

Mathematics Subject Classification: xxxxx

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1 Introduction

The notion of function of bounded variation, or BV function, was introduced by C. Jordan in 1881, [9], when he critically re-examined a faulty proof given by Dirichlet to the famous Fourier's conjecture on trigonometric series expansion of periodic functions, see [11]. By showing that functions of bounded variation are precisely those that can be expressed as the difference of two monotone functions, Jordan actually extended the celebrated Dirichlet's criterium to the class of BV functions. Since then, the notion has been generalized in several ways leading to many important results in mathematical analysis.

Two well-known generalizations are the functions of bounded p-variation and functions of bounded ϕ-variation introduced by N. Wiener [21] and L. C. Young [22] respectively. Many basic properties of the variation in the sense of Wiener and a large number of important applications of this concept can be found in [1] and [5].

We denote by $BV([a,b]) = BV([a,b],\mathbb{R})$ the space of all real-valued functions of bounded variation on the interval $[a,b]$.

We will say that a sequence of positive real numbers, $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$, is a $\mathcal{W}$-sequence if it is non-decreasing and $\sum(1/\lambda_i) = +\infty$.

In 1972, D. Waterman [20] introduced the class $\Lambda BV([a,b])$ of functions of bounded Λ-variation and in 1980, M. Shiba [17] generalized Waterman's notion by introducing the class $\Lambda_p BV([a,b])$ $(1 \le p < \infty)$; that is, the class of all functions $f : [a,b] \to \mathbb{R}$ with bounded Λ_p-variation on $[a,b]$. Following:

Definition 1.1 ([19]). *Let $I = [a,b] \subset \mathbb{R}$ be a closed interval. Let Λ be a $\mathcal{W}$-sequence and suppose that $p \ge 1$. A function $f : I \to \mathbb{R}$ is said to be of bounded Λ_p-variation on I ($f \in \Lambda_p BV(I)$) if*

$$V_{\Lambda_p}(f) = V_\Lambda(f,p,I) = \sup_{\xi} V_\Lambda(\xi, f, p, I) < \infty,$$

where

$$V_\Lambda(\xi, f, p, I) := \left(\sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{\lambda_i} \right)^{1/p},$$

and the supremum is taken over all partitions $\xi : a = x_0 < x_1 < \cdots < x_n = b$ of the interval I.

In more recent years, there has been a growing interest in the study of various mathematical problems concerning classes of functions whose growth is limited by considering variable exponents, such as the, so called, variable L^p spaces (or variable exponent $L^{p(\cdot)}$), especially for its applications in the theory of non-newtonian fluids and image processing, see e.g. [12], [15], [23]. Lebesgue spaces with variable exponent have been introduced already in 1931 by Orlicz, see [4], [14] and [16]. In [10] the authors established many of the basic properties of Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev spaces $W^{k,p(x)}$. In [2], Castillo, Merentes and Rafeiro studied a new space of functions of bounded $p(\cdot)$-variation, they introduced the notion of bounded variation in the sense of Wiener with exponent $p(\cdot)$-variable. Letter, in [13], Mejía, Merentes and Sánchez characterized the functions of those spaces, establishing important properties and also they presented some properties of the composition (Nemytskij) operator when it acts between such spaces.

2 Space of Functions of Bounded $\Lambda_{p(\cdot)}$-Variation

Here we introduce the notion of variation in the Waterman-Shiba's sense with variable exponent.

Definition 2.1. *Let $[a, b] \subset \mathbb{R}$ be a closed interval and let Λ be a $\mathcal{W}$-sequence. Given a function $p : [a, b] \to (1, +\infty)$ and a function $f : [a, b] \to \mathbb{R}$, the functional $V_\Lambda^{p(\cdot)}(f)$ given by*

$$V_\Lambda^{p(\cdot)}(f) = V_\Lambda^{p(\cdot)}(f; [a, b]) := \sup_{\pi^*} \sum_{i=1}^{n} \frac{|f(t_i) - f(t_{i-1})|^{p(x_{i-1})}}{\lambda_i} \tag{1}$$

will be called the $\Lambda_{p(\cdot)}$-variation of f on $[a, b]$, where the supremum is taken on all tagged partitions π^* of $[a, b]$; i.e., a partition of the interval $[a, b]$ together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ such that $t_j \leq x_j \leq t_{j+1}$ for all $j = 0, 1, \ldots, n-1$.

Note that any partition $\pi = \{x_0, \ldots, x_n\}$ also can be viewed as a union of non-overlapping intervals $\bigcup_{i=1}^{n} I_i$ with $I_i : [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Definition 2.2. *The set of functions of bounded $\Lambda_{p(\cdot)}$-variation in the Waterman-Shiba's sense is defined as*

$$\Lambda_{p(\cdot)}BV([a, b]) = \left\{ f : [a, b] \to \mathbb{R} / \; \exists\, \beta > 0 \text{ where } V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) < \infty \right\}.$$

Remark 2.3.

1. *If* $p(x) = 1$ *for all* $x \in [a, b]$ *then*

$$\Lambda_{p(\cdot)}BV([a, b]) = \Lambda BV([a, b]).$$

2. *If* $p(x) = p,$ *for all* $x \in [a, b]$ *with* $1 < p < +\infty$ *then*

$$V_{\Lambda}^{p(\cdot)}(f) = [V_{\Lambda_p}(f)]^p.$$

Lemma 2.4. *(General Properties of the* $\Lambda_{p(\cdot)}$*-variation) Let* Λ *be a* $\mathcal{W}$*-sequence,* $p : [a, b] \to [1, +\infty)$ *a function and let* $f : [a, b] \to \mathbb{R}$ *be a function.*

(P1) Minimality: If $s, t \in [a, b]$ *with* $s < t$*, then*

$$\frac{|f(t) - f(s)|^{p(x_{st})}}{\lambda_{st}} \leq V_{\Lambda}^{p(\cdot)}(f).$$

where x_{ts} *a number between* $[t, s]$ *and* λ_{st} *is the corresponding term in the sequence* Λ *associated to any partition* π *that contains* $[s, t]$ *as subinterval.*

(P2) Monotonicity: If $s, t \in [a, b]$ *with* $a \leq s \leq t \leq b$*, then* $V_{\Lambda}^{p(\cdot)}(f; [a, s]) \leq V_{\Lambda}^{p(\cdot)}(f; [a, t]),\quad V_{\Lambda}^{p(\cdot)}(f; [t, b]) \leq V_{\Lambda}^{p(\cdot)}(f; [s, b])$ *and*

$$V_{\Lambda}^{p(\cdot)}(f; [s, t]) \leq V_{\Lambda}^{p(\cdot)}(f; [a, b]).$$

(P3) Semi-additivity: If $c \in [a, b]$ *then*

$$V_{\Lambda}^{p(\cdot)}(f; [a, b]) \geq V_{\Lambda}^{p(\cdot)}(f; [a, c]) + V_{\Lambda}^{p(\cdot)}(f; [c, b]).$$

(P4) Change of variable: If $[c, d] \subset \mathbb{R}$ *and* $\varphi : [c, d] \to [a, b]$ *is a monotone function (not necessarily strictly), then*

$$V_{\Lambda}^{p(\cdot)}(f; \varphi([c, d])) = V_{\Lambda}^{p(\cdot)}(f \circ \varphi; [c, d]).$$

(P5) Regularity:

$$V_{\Lambda}^{p(\cdot)}(f; [a, b]) = \sup\left\{V_{\Lambda}^{p(\cdot)}(f; [s, t]); s, t \in [a, b], s \leq t\right\}.$$

Proof. The proof of these properties are obtained by using arguments similar to those presented in Lemma 2 in [13].

$\square$

3 Further Properties of Functions of Bounded $\Lambda_{p(\cdot)}$-Variation

Here we present some further properties of the functionals $V_\Lambda^{p(\cdot)}(\cdot)$ that we will need later to prove that the set $\Lambda_{p(\cdot)}BV([a,b])$ actually is a vector space.

Theorem 3.1. *If Λ is a $\mathcal{W}$-sequence, $p:[a,b]\to(1,+\infty)$ is a function and $f:[a,b]\to\mathbb{R}$ is a function then*

1. $V_\Lambda^{p(\cdot)}(|f|)\le V_\Lambda^{p(\cdot)}(f)$ *for all* $f\in\Lambda_{p(\cdot)}BV([a,b])$.
2. *If* $\beta_1>\beta_2$ *then* $V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_1}\right)\le V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_2}\right)$ *for all* $f\in\Lambda_{p(\cdot)}BV([a,b])$.
3. $V_\Lambda^{p(\cdot)}$ *is convex.*

Proof. Let $\pi: a=t_0<t_1<\cdots<t_n=b$ a partition of the interval $[a,b]$ and let π^* be a tagged partition of π. Then

1. It readily follows using that
$$||f(t_i)|-|f(t_{i-1})||\le|f(t_i)-f(t_{i-1})|.$$
Note also that if $f\in\Lambda_{p(\cdot)}BV([a,b])$ then $|f|\in\Lambda_{p(\cdot)}BV([a,b])$.
2. Let β_1,β_2 such that $\beta_1>\beta_2$. Then,
$$\begin{aligned}\left|\left(\frac{f}{\beta_1}\right)(t_i)-\left(\frac{f}{\beta_1}\right)(t_{i-1})\right|^{p(x_{i-1})} &= \left(\frac{1}{\beta_1}|f(t_i)-f(t_{i-1})|\right)^{p(x_{i-1})}\\ &\le \left(\frac{1}{\beta_2}|f(t_i)-f(t_{i-1})|\right)^{p(x_{i-1})}\\ &= \left|\left(\frac{f}{\beta_2}\right)(t_i)-\left(\frac{f}{\beta_2}\right)(t_{i-1})\right|^{p(x_{i-1})}.\end{aligned}$$
Dividing by λ_i and taking supremum over all tagged partitions π^* we obtain
$$V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_1}\right)\le V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_2}\right).$$
3. Let $\alpha,\beta\ge 0$ such that $\alpha+\beta=1$, and $f,g\in\Lambda_{p(\cdot)}BV([a,b])$. Using the convexity of the function $h(t)=t^a$, $a\ge1$ we have
$$\begin{aligned}&|(\alpha f+\beta g)(t_i)-(\alpha f+\beta g)(t_{i-1})|^{p(x_{i-1})}\\ &\quad= |\alpha(f(t_i)-f(t_{i-1}))+\beta(g(t_i)-g(t_{i-1}))|^{p(x_{i-1})}\\ &\quad\le (\alpha|f(t_i)-f(t_{i-1})|+\beta|g(t_i)-g(t_{i-1})|)^{p(x_{i-1})}\\ &\quad\le \alpha|f(t_i)-f(t_{i-1})|^{p(x_{i-1})}+\beta|g(t_i)-g(t_{i-1})|^{p(x_{i-1})}.\end{aligned}$$

Thus

$$\frac{|(\alpha f+\beta g)(t_i)-(\alpha f+\beta g)(t_{i-1})|^{p(x_{i-1})}}{\lambda_i} \leq \alpha\frac{|f(t_i)-f(t_{i-1})|^{p(x_{i-1})}}{\lambda_i}+\beta\frac{|g(t_i)-g(t_{i-1})|^{p(x_{i-1})}}{\lambda_i}.$$

Taking supremum over all tagged partitions π^* we finally obtain:

$$V_\Lambda^{p(\cdot)}(\alpha\, f+\beta\, g)\leq \alpha\, V_\Lambda^{p(\cdot)}(f)+\beta\, V_\Lambda^{p(\cdot)}(g).$$

□

Now we prove that the set of all functions of bounded $\Lambda_{p(\cdot)}$-variation in the Waterman-Shiba's sense, on an interval $[a,b]$, is a vector space.

Theorem 3.2. *$\Lambda_{p(\cdot)}BV([a,b])$ is a vector space.*

Proof. Let $f,g\in\Lambda_{p(\cdot)}BV([a,b])$ and suppose $\alpha,\beta\in\mathbb{R}$. Then, there are $\beta_1,\beta_2>0$ such that

$$V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_1}\right)<\infty \quad \text{and} \quad V_\Lambda^{p(\cdot)}\left(\frac{g}{\beta_2}\right)<\infty.$$

Let $\hat\beta:=\max\{\beta_1,\beta_2\}>0$. By property (2) of Theorem 3.1 we have

$$V_\Lambda^{p(\cdot)}\left(\frac{f}{\hat\beta}\right)\leq V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_1}\right)<\infty$$

and

$$V_\Lambda^{p(\cdot)}\left(\frac{f}{\hat\beta}\right)\leq V_\Lambda^{p(\cdot)}\left(\frac{g}{\beta_2}\right)<\infty.$$

If $\alpha=\beta=0$, $\alpha f+\beta g=0\in\Lambda_{p(\cdot)}BV([a,b])$. We suppose then that $\alpha\neq 0\vee\beta\neq 0$.

Let $\mu=(|\alpha|+|\beta|)\,\hat\beta>0$ and let π^* be a tagged partition of $[a,b]$. Given $f,g\in\Lambda_{p(\cdot)}BV([a,b])$ and $i\in\{1,2,\ldots,n\}$, we have

$$\begin{aligned}|(\alpha f+\beta g)(t_i)-(\alpha f+\beta g)(t_{i-1})| &= |\alpha(f(t_i)-f(t_{i-1}))+\beta(g(t_i)-g(t_{i-1}))|\\ &\leq |\alpha||f(t_i)-f(t_{i-1})|+|\beta||g(t_i)-g(t_{i-1})|.\end{aligned}$$

Thus,

$$\left|\left(\frac{\alpha f+\beta g}{\mu}\right)(t_i)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{i-1})\right|\leq\frac{|\alpha|}{\mu}|f(t_i)-f(t_{i-1})|+\frac{|\beta|}{\mu}|g(t_i)-g(t_{i-1})|,$$

which imply that

$$
\begin{aligned}
&\left|\left(\frac{\alpha f+\beta g}{\mu}\right)(t_i)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{i-1})\right|^{p(x_{i-1})}\\
&\quad\leq\left(\frac{|\alpha|}{|\alpha|+|\beta|}\frac{|f(t_i)-f(t_{i-1})|}{\hat{\beta}}+\frac{|\beta|}{|\alpha|+|\beta|}\frac{|g(t_i)-g(t_{i-1})|}{\hat{\beta}}\right)^{p(x_{i-1})}\\
&\quad\leq\frac{|\alpha|}{|\alpha|+|\beta|}\left(\frac{|f(t_i)-f(t_{i-1})|}{\hat{\beta}}\right)^{p(x_{i-1})}+\frac{|\beta|}{|\alpha|+|\beta|}\left(\frac{|g(t_i)-g(t_{i-1})|}{\hat{\beta}}\right)^{p(x_{i-1})}\\
&\quad=\frac{|\alpha|}{|\alpha|+|\beta|}\left|\left(\frac{f}{\hat{\beta}}\right)(t_i)-\left(\frac{f}{\hat{\beta}}\right)(t_{i-1})\right|^{p(x_{i-1})}\\
&\qquad+\frac{|\beta|}{|\alpha|+|\beta|}\left|\left(\frac{g}{\hat{\beta}}\right)(t_i)-\left(\frac{g}{\hat{\beta}}\right)(t_{i-1})\right|^{p(x_{i-1})}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\left|\left(\frac{\alpha f+\beta g}{\mu}\right)(t_i)-\left(\frac{\alpha f+\beta g}{\mu}\right)(t_{i-1})\right|^{p(x_{i-1})}}{\lambda_i}&\leq\frac{|\alpha|}{|\alpha|+|\beta|}\frac{\left|\left(\frac{f}{\hat{\beta}}\right)(t_i)-\left(\frac{f}{\hat{\beta}}\right)(t_{i-1})\right|^{p(x_{i-1})}}{\lambda_i}\\
&\quad+\frac{|\beta|}{|\alpha|+|\beta|}\frac{\left|\left(\frac{g}{\hat{\beta}}\right)(t_i)-\left(\frac{g}{\hat{\beta}}\right)(t_{i-1})\right|^{p(x_{i-1})}}{\lambda_i}.
\end{aligned}
$$

Taking supremum over all tagged partitions π^* of $[a,b]$ we get

$$
V_\Lambda^{p(\cdot)}\left(\frac{\alpha\, f+\beta\, g}{\mu}\right)\leq\frac{|\alpha|}{|\alpha|+|\beta|}V_\Lambda^{p(\cdot)}\left(\frac{f}{\hat{\beta}}\right)+\frac{|\alpha|}{|\alpha|+|\beta|}V_\Lambda^{p(\cdot)}\left(\frac{g}{\hat{\beta}}\right)<\infty.
$$

Therefore,

$$
\alpha\, f+\beta\, g\in\Lambda_{p(\cdot)}BV([a,b]).
$$

We conclude that $\Lambda_{p(\cdot)}BV([a,b])$ is indeed a vector space.

$\square$

Now we define the function

$$
\|\cdot\|_{\Lambda_{p(\cdot)}}:\Lambda_{p(\cdot)}BV([a,b])\to\mathbb{R}
$$

by

$$
\|f\|_{\Lambda_{p(\cdot)}}:=|f(a)|+\inf\left\{\beta>0:V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right)\leq 1\right\}
$$

for each $f\in\Lambda_{p(\cdot)}BV([a,b])$.

We now prove that $\|\cdot\|$ defines a norm on $\Lambda_{p(\cdot)}BV([a,b])$.

Theorem 3.3. *$(\Lambda_{p(\cdot)}BV([a,b]); \|\cdot\|_{\Lambda_{p(\cdot)}})$ is a normed vector space.*

Proof. Let $f, g \in \Lambda_{p(\cdot)}BV([a,b])$, $\alpha \in \mathbb{R}$ and let $\Lambda = \{\lambda_i\}_{i\geq 1}$ be a W-sequence.

1. Clearly by definition:

$$\|f\|_{\Lambda_{p(\cdot)}} \geq 0, \text{ for all } f \in \Lambda_{p(\cdot)}BV([a,b]).$$

2. If $\alpha = 0$ then $\|\alpha\, f\|_{\Lambda_{p(\cdot)}} = |\alpha|\, \|f\|_{\Lambda_{p(\cdot)}}$.
   Suppose that $\alpha \neq 0$. Then,

$$\begin{aligned}
\|\alpha\, f\|_{\Lambda_{p(\cdot)}} &= |\alpha\, f(a)| + \inf\left\{\beta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{\alpha\, f}{\beta}\right) \leq 1\right\} \\
&= |\alpha|\, |f(a)| + \inf\left\{\beta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\frac{\beta}{|\alpha|}}\right) \leq 1\right\} \\
&= |\alpha|\, |f(a)| + |\alpha| \inf\left\{\frac{\beta}{|\alpha|} > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\frac{\beta}{|\alpha|}}\right) \leq 1\right\} \\
&= |\alpha| \left(|f(a)| + \inf\left\{\hat{\beta} > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\hat{\beta}}\right) \leq 1\right\}\right) \\
&= |\alpha| \|f\|_{\Lambda_{p(\cdot)}}.
\end{aligned}$$

3. Let $\beta_1, \beta_2 > 0$ such that

$$\beta_1 > \inf\left\{\beta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\beta}\right) \leq 1\right\} := C_1$$

   and

$$\beta_2 > \inf\left\{\beta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{g}{\beta}\right) \leq 1\right\} := C_2.$$

   Thus, by the definition of infimum, there are $\hat{\beta}_1, \hat{\beta}_2$ such that

$$C_1 < \hat{\beta}_1 < \beta_1 \quad \text{with} \quad V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\hat{\beta}_1}\right) \leq 1$$

   and

$$C_2 < \hat{\beta}_2 < \beta_2 \quad \text{with} \quad V_{\Lambda}^{p(\cdot)}\left(\frac{g}{\hat{\beta}_2}\right) \leq 1,$$

   then, using property (2) of Theorem 3.1, we have

$$V_{\Lambda}^{p(\cdot)}\left(\frac{f}{\beta_1}\right) \leq 1 \quad \text{and} \quad V_{\Lambda}^{p(\cdot)}\left(\frac{g}{\beta_2}\right) \leq 1.$$

Put $\hat{\beta} := \beta_1 + \beta_2$. As $V_\Lambda^{p(\cdot)}(f)$ is convex

$$\begin{aligned} V_\Lambda^{p(\cdot)}\left(\frac{f+g}{\hat{\beta}}\right) &= V_\Lambda^{p(\cdot)}\left(\frac{\beta_1}{\hat{\beta}}\frac{f}{\beta_1} + \frac{\beta_2}{\hat{\beta}}\frac{g}{\beta_2}\right) \\ &\leq \frac{\beta_1}{\beta_1+\beta_2} V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta_1}\right) + \frac{\beta_1}{\beta_1+\beta_2} V_\Lambda^{p(\cdot)}\left(\frac{g}{\beta_2}\right) \\ &\leq 1, \end{aligned}$$

consequently,

$$\begin{aligned} \|f+g\|_{\Lambda_{p(\cdot)}} &= |(f+g)(a)| + \inf\left\{\delta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f+g}{\delta}\right) \leq 1\right\} \\ &\leq |f(a)+g(a)| + \hat{\beta} \\ &\leq |f(a)| + \beta_1 + |g(a)| + \beta_2, \end{aligned}$$

since β_1 and β_2 are arbitrary, particularly for $\beta_1 = \beta_2 := 1/2n$ with $n \in \mathbb{N}$ we obtain

$$\begin{aligned} \|f+g\|_{\Lambda_{p(\cdot)}} &\leq |f(a)| + C_1 + \frac{1}{2n} + |g(a)| + C_2 + \frac{1}{2n} \\ &= |f(a)| + C_1 + |g(a)| + C_2 + \frac{1}{n}, \end{aligned}$$

and taking limit when $n \to +\infty$, we obtain

$$\begin{aligned} \|f+g\|_{\Lambda_{p(\cdot)}} &\leq |f(a)| + C_1 + |g(a)| + C_2 \\ &= \|f\|_{\Lambda_{p(\cdot)}} + \|g\|_{\Lambda_{p(\cdot)}}. \end{aligned}$$

4. We now prove that $\|f\|_{\Lambda_{p(\cdot)}} = 0$ if and only if $f = 0$.

Note that if $f = 0$, $V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) = 0 \leq 1$ for all $\beta > 0$, so that

$$\inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) \leq 1\right\} = 0$$

and since $f(a) = 0$ we obtain $\|f\|_{\Lambda_{p(\cdot)}} = 0$. Suppose now that $\|f\|_{\Lambda_{p(\cdot)}} = 0$. Then

$$|f(a)| + \inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) \leq 1\right\} = 0,$$

which implies that

$$|f(a)| = 0 \quad \text{and} \quad \inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) \leq 1\right\} = 0.$$

Now, if $\inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) \le 1\right\} = 0$, given $\varepsilon > 0$, there exists $0 < \beta < \varepsilon$ such that

$$V_\Lambda^{p(\cdot)}\left(\frac{f}{\varepsilon}\right) \le V_\Lambda^{p(\cdot)}\left(\frac{f}{\beta}\right) \le 1.$$

This implies that $V_\Lambda^{p(\cdot)}\left(\frac{f}{\varepsilon}\right) \le 1$, for all $\varepsilon > 0$, particularly for $0 < \varepsilon < 1$,

$$\begin{aligned} V_\Lambda^{p(\cdot)}(f) = V_\Lambda^{p(\cdot)}\left(\varepsilon\,\frac{f}{\varepsilon}\right) &= V_\Lambda^{p(\cdot)}\left(\varepsilon\,\frac{f}{\varepsilon} + (1-\varepsilon)\,0\right) \\ &\le \varepsilon\, V_\Lambda^{p(\cdot)}\left(\frac{f}{\varepsilon}\right) + (1-\varepsilon)\, V_\Lambda^{p(\cdot)}(0) \\ &\le \varepsilon. \end{aligned}$$

Thus, $0 \le V_\Lambda^{p(\cdot)}(f) \le \varepsilon$, for all $\varepsilon > 0$ and this implies that

$$V_\Lambda^{p(\cdot)}(f) = 0.$$

Moreover, without loss of generality we may consider the partition $\pi : a = t_0 < t_1 = x < t_2 = b$, from which we obtain

$$\frac{|f(x) - f(a)|^{p(a)}}{\lambda_1} + \frac{|f(b) - f(x)|^{p(x)}}{\lambda_2} = 0,$$

then $|f(x) - f(a)| = 0$ and $|f(b) - f(x)| = 0$ so that $f(x) = f(a) = f(b)$, for all $x \in [a, b]$ and as $f(a) = 0$ we obtain $f = 0$.

□

Theorem 3.4. *Let $p : [a, b] \to [1, +\infty)$ be a function and let Λ be a W-sequence, then $(\Lambda_{p(\cdot)}BV([a, b]); \|\cdot\|_{\Lambda_{p(\cdot)}})$ is a Banach's space.*

Proof. Let $\Lambda = \{\lambda_i\}_{i\ge1}$ a W-sequence and suppose that $\{f_n\}_{n\ge1}$ is a Cauchy's sequence in $(\Lambda_{p(\cdot)}BV([a, b]); \|\cdot\|_{\Lambda_{p(\cdot)}})$. Then, given $\varepsilon > 0$, there exists $N > 0$ such that for $m, n \ge N$ we obtain

$$\|f_n - f_m\|_{\Lambda_{p(\cdot)}} \le \frac{\varepsilon}{2},$$

that is to say,

$$|f_n(a) - f_m(a)| + \inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f_n - f_m}{\beta}\right) \le 1\right\} < \frac{\varepsilon}{2}.$$

Thus $|f_n(a) - f_m(a)| < \dfrac{\varepsilon}{2}$ and

$$\inf\left\{\beta > 0 : V_\Lambda^{p(\cdot)}\left(\frac{f_n - f_m}{\beta}\right) \le 1\right\} < \frac{\varepsilon}{2},\ n, m \ge N;$$

then, by the definition of infimum and the property 2 of Theorem 3.1,

$$V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f_m}{\varepsilon/2}\right) \leq 1.$$

Let us fix $s, t \in [a, b]$ with $s < t$ and consider, without loss of generality, the tagged partition $\pi^* : a = t_0 < x_0 < t_1 = s < x_1 < t_2 = t < x_2 < t_3 = b$. Let $M = \max\{\lambda_2, 1\}$ be, then

$$\begin{aligned}
\frac{\left(\frac{|(f_n-f_m)(t_2)-(f_n-f_m)(t_1)|}{\varepsilon/2}\right)^{p(x_1)}}{\lambda_2} &\leq \sum_{i=1}^{3} \frac{\left(\frac{|(f_n-f_m)(t_i)-(f_n-f_m)(t_{i-1})|}{\varepsilon/2}\right)^{p(x_{i-1})}}{\lambda_i} \\
&\leq V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f_m}{\varepsilon/2}\right) \leq 1
\end{aligned}$$

therefore,

$$|(f_n - f_m)(t_2) - (f_n - f_m)(t_1)|^{p(x_1)} \leq \left(\frac{\varepsilon}{2}\right)^{p(x_1)} \lambda_2 \leq \left(\frac{\varepsilon}{2}\right)^{p(x_1)} M$$

hence

$$|(f_n - f_m)(t) - (f_n - f_m)(s)|^{p(x_1)} \leq \left(\frac{\varepsilon}{2}\, M\right)^{p(x_1)}$$

thus

$$|(f_n - f_m)(t) - (f_n - f_m)(s)| \leq M\,\frac{\varepsilon}{2}, \quad \text{for all } s, t \in [a, b].$$

Let us fix $s = a$,

$$|f_n(t) - f_m(t) - (f_n(a) - f_m(a))| \leq M\,\frac{\varepsilon}{2}, \quad \text{for all } t \in [a, b]$$

and as $|f_n(a) - f_m(a)| \leq \varepsilon/2$ we obtain

$$\begin{aligned}
|f_n(t) - f_m(t)| &= |f_n(t) - f_n(a) + f_n(a) + f_m(a) - f_m(a) - f_m(t)| \\
&\leq |f_n(t) - f_m(t) - (f_n(a) + f_m(a))| + |f_n(a) - f_m(a)| \\
&\leq M\,\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{(M+1)\,\varepsilon}{2}, \quad \text{for all } \ t \in [a, b].
\end{aligned}$$

Therefore, $\{f_n\}_{n\geq 1}$ is a uniform Cauchy sequence on the interval $[a, b]$. Since $\mathbb{R}$ is complete, there is a function f defined on $[a, b]$ such that

$$\lim_{n\to+\infty} f_n(t) = f(t), \quad t \in [a, b].$$

We now prove that $\{f_n\}_{n\geq 1}$ converges in the norm $\|\cdot\|_{\Lambda_{p(\cdot)}}$. Note that

$$\|f_n - f\|_{\Lambda_{p(\cdot)}} = |f_n(a) - f(a)| + \inf\left\{\beta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f}{\beta}\right) \leq 1\right\}. \tag{2}$$

Recall that $|f_n(a) - f_m(a)| < \varepsilon/2$. Taking limit when $m \to +\infty$ we have

$$|f_n(a) - f(a)| \leq \varepsilon/2, \quad n \geq N. \tag{3}$$

On the other hand, consider the tagged partition

$$\pi^* : a = t_0 < x_0 < t_1 < \cdots < t_{k-1} < x_{k-1} < t_k = b$$

and let us fix $n \geq N$. Then,

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{\left(\dfrac{|f_n(t_i) - f(t_i) - (f_n(t_{i-1}) - f(t_{i-1}))|}{\varepsilon/2}\right)^{p(x_{i-1})}}{\lambda_i} \\
&\quad = \lim_{m\to+\infty} \sum_{i=1}^{k} \frac{\left(\dfrac{|f_n(t_i) - f_m(t_i) - (f_n(t_{i-1}) - f_m(t_{i-1}))|}{\varepsilon/2}\right)^{p(x_{i-1})}}{\lambda_i} \\
&\quad \leq \limsup_{m\to+\infty} V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f_m}{\varepsilon/2}\right) \leq 1,
\end{aligned}$$

this implies that $V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f}{\varepsilon/2}\right) \leq 1$ and therefore

$$\inf\left\{\delta > 0 : V_{\Lambda}^{p(\cdot)}\left(\frac{f_n - f}{\delta}\right) \leq 1\right\} \leq \frac{\varepsilon}{2}. \tag{4}$$

Returning to equation (2) and we using the results (3) and (4) we obtain

$$\|f_n - f\|_{\Lambda_{p(\cdot)}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad n \geq N.$$

Thus, the sequence $\{f_n\}_{n\geq 1}$ converge to the function f in the norm $\|\cdot\|_{\Lambda_{p(\cdot)}}$ and in this manner we have to

$$(\Lambda_{p(\cdot)}BV([a,b]);\ \|\cdot\|_{\Lambda_{p(\cdot)}})$$

is a Banach space.

□

4 Characterization of $\Lambda_{p(\cdot)}BV([a,b])$ by Compositions

A problem that has attracted the attention of many mathematicians has been the representation "by the composition" of certain basic functions the elements of new classes, or existing classes of functions; i.e., to characterize somehow the functions of a given space as the composition of two simpler functions. We begin by mentioning that W. Sierpinski in 1933 (see [18]) showed that a function $f : [a,b] \to \mathbb{R}$ is *regulated* if, and only if, it is the composition of an increasing function and a continuous function; then, H. Federer in 1969 (see [7]) showed that a function is of bounded variation if, and only if, it is the composition of a Lipschitz function with a monotone function. Chistyakov and Galkin in 1998 (see [3]), proved a similar result for functions of p-bounded variation with $p > 1$: they showed that a function is of p-bounded variation if, and only, if it is the composition is a non-decreasing bounded function with a Hölder function.

Let $p : [a,b] \to [1,+\infty)$ be a function and let $\gamma(\cdot)$ be a function such that $0 < \gamma(\cdot) \leq 1$. We say that $g \in H^{\gamma(\cdot)}$, the Hölder space of variable exponent, if there exists $C > 0$ such that

$$|g(t_i) - g(t_{i-1})| \leq C\ |t_i - t_{i-1}|^{\gamma(x_{i-1})}, \quad \text{for all } x_{i-1} \in [a,b].$$

The least number C satisfying the above inequality is called the Hölder constant of g.

Following these ideas, first we exhibit present next a procedure, involving compositions, to produce plenty of concrete examples of functions of $p(\cdot)$-variation in the sense of Waterman-Shiba.

Theorem 4.1. *If $\varphi : [a,b] \to \mathbb{R}$ is monotone and bounded, $g : \varphi([a,b]) \to \mathbb{R}$ is a Hölder function with variable exponent $\gamma(\cdot) = \dfrac{1}{p(\cdot)}$ and $f = g \circ \varphi$, then $f \in \Lambda_{p(\cdot)}BV([a,b])$.*

Proof. Let Λ a W-sequence. Suppose that φ is non-decreasing (the decreasing case is treated analogously). Then, $\varphi([a,b]) = [\varphi(a), \varphi(b)]$ and by property 4 of Lemma 2.4 we have

$$V_\Lambda^{p(\cdot)}\left(\frac{f}{C}; [a,b]\right) = V_\Lambda^{p(\cdot)}\left(\frac{g\circ\varphi}{C}; [a,b]\right) = V_\Lambda^{p(\cdot)}\left(\frac{g}{C}; \varphi([a,b])\right) \text{ for all } C > 0. \tag{5}$$

Since $g \in H^{\gamma(\cdot)}$, there exists $C > 0$ such that

$$|g(t_i) - g(t_{i-1})| \leq C\ |t_i - t_{i-1}|^{\gamma(x_{i-1})} \quad \text{for all } x_{i-1} \in [a,b],$$

this implies that

$$\frac{|g(t_i) - g(t_{i-1})|}{C} \leq |t_i - t_{i-1}|^{\gamma(x_{i-1})}$$

then,

$$\left[\frac{|g(t_i) - g(t_{i-1})|}{C}\right]^{p(x_{i-1})} \leq |t_i - t_{i-1}|^{p(x_{i-1})\,\gamma(x_{i-1})} = |t_i - t_{i-1}|$$

so that

$$\frac{\left[\frac{|g(t_i)-g(t_{i-1})|}{C}\right]^{p(x_{i-1})}}{\lambda_i} \leq \frac{|t_i - t_{i-1}|}{\lambda_i} \leq \frac{|t_i - t_{i-1}|}{\lambda_1}. \tag{6}$$

Consider now $T = \{t_i\}_{i=1}^m$ a partition of $[\varphi(a), \varphi(b)]$, using (6) we have

$$\sum_{i=1}^{m} \frac{\left[\frac{|g(t_i) - g(t_{i-1})|}{C}\right]^{p(x_{i-1})}}{\lambda_i} \leq \sum_{i=1}^{m} \frac{|t_i - t_{i-1}|}{\lambda_1} \leq \frac{\varphi(b) - \varphi(a)}{\lambda_1},$$

now, we take supreme on T and remembering that φ is bounded, we obtain

$$V_{\Lambda}^{p(\cdot)}\left(\frac{g}{C}; T\right) \leq \frac{\varphi(b) - \varphi(a)}{\lambda_1} < +\infty.$$

This implies that

$$V_{\Lambda}^{p(\cdot)}\left(\frac{g}{C}; \varphi([a,b])\right) = V_{\Lambda}^{p(\cdot)}\left(\frac{g}{C}; [\varphi(a), \varphi(b)]\right) < +\infty.$$

Thus, returning to equation (5) we have

$$V_{\Lambda}^{p(\cdot)}\left(\frac{f}{C}; \varphi([a,b])\right) < +\infty.$$

Therefore, $f = g \circ \varphi \in \Lambda_{p(\cdot)}BV([a,b])$.

$\square$

Next we present the following result that states that $\Lambda_{p(\cdot)}BV$ is invariant under monotone substitutions of variables this concerns with the theory of linear composition operators:

Proposition 4.2. *Given a function $g : [c,d] \to \mathbb{R}$, let $\tau : [a,b] \to [c,d]$ be continuous and strictly increasing function with $\tau(a) = c$ and $\tau(b) = d$. Then $g \circ \tau \in \Lambda_{p(\cdot)}BV[a,b]$ if and only if $g \in \Lambda_{p(\cdot)}BV[c,d]$.*

Proof. Let $\Pi([a,b])$ denote the set of all partitions of the interval $[a,b]$ and let $\pi^* = \{x_0, x_1, \ldots, x_n\} \in [a,b]$, then $\tau\pi^* := \{\tau(x_0), \tau(x_1), \ldots, \tau(x_n)\} \in \Pi([c,d])$ and we have $\Pi[a,b] = \tau(\Pi([c,d]))$. The result follows by using a similar argument as the one given in the proof of Proposition 2.22 in [1].

□

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