Finite – Time Ruin Probability
In a Generalized Risk Processes under Interest Force

Bui Khoi Dam
Applied Mathematics and Informatics School
Hanoi University of Science and Technology, Ha noi, Viet Nam
Phung Duy Quang
Foreign Trade University, Ha noi, Viet Nam
Email: quangmathftu@yahoo.com

Abstract

The aim of this paper is to build an exact formula for ruin probability of generalized risk processes under interest force with assumption that claims and premiums are assumed to be positive-valued random variables and interests are assumed to be non-negative-valued random variables (claims, premiums and interests are assumed to be independent). This situation is quite realistic for many situations. An exact formula for ruin (non-ruin) probabilities is derived in this paper. A numerical example is given to illustrate results. Our results is to extend models which is an exact formula derived by Claude Lefèvre and Stéphane Loisel [6].

Mathematics Subject Classifications. 62P05,60G40, 12E05

Key words. Ruin probability, Non- Ruin probability.

1. Introduction

For over a century, there has been a major interest in actuarial science. Since a large portion of the surplus of insurance business from investment income, actuaries have been studying ruin problems under risk models with rates of interest. For example, Teugels and Sundt [20], [21] studied the effects of constant rate on the ruin probability under the compound Poisson risk model. Yang [23] established both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Cai [3], [4] investigated the ruin probabilities in two risk models, with independent premiums

1 Corresponding author: Phung Duy Quang
and claims and used a first-order autoregressive process to model the rates of in interest. Cai
and Dickson [5] obtained Lundberg inequalities for ruin probabilities in two discrete-time risk
process with a Markov chain interest model and independent premiums and claims. However,
those results is only given upper bounds for finite-time probabilities and ultimate ruin
probability that they did not provide an exact formula for finite-time probabilities.
Claude Lefèvre and Stéphane Loisel [6] studied the problem of ruin in the classical compound
binomial and compound Poisson risk models. Their primary purpose is to extend those models
which is an exact formula derived by Pircard and Lefèvre [7] for the probability of (non-ruin)
ruin within finite time.
However, Claude Lefèvre and Stéphane Loisel [6] did not provide an exact formula for ruin
probability of generalized risk processes under interest force with surplus process \( \{U_{t}\}_{t \geq 0} \)
written as
\[
U_t = U_{t-1}(1 + I_t) + X_t - Y_t; t = 1, 2, \ldots
\]
(1.1)
or
\[
U_t = (U_{t-1} + X_t)(1 + I_t) - Y_t; t = 1, 2, \ldots
\]
(1.2)
where \( U_0 = u \) is initial surplus, \( u \) and \( t \) are positive integer numbers, \( X = \{X_{t}\}_{t \geq 0} \) and
\( Y = \{Y_{j}\}_{j \geq 1} \) take values in a finite set of positive numbers; \( I = \{I_{k}\}_{k \geq 1} \) take values in a finite set
of non-negative numbers. \( X \), \( Y \) and \( I \) are assumed to be independent.
The aim of this paper is to build an exact formula for finite time ruin (non-ruin) probability of
model (1.1) and (1.2) with these assumptions. We establish an exact formula for ruin (non-ruin)
probability of model (1.1) and (1.2) whose exact formula for finite time ruin (non-ruin)
probability are derived.
The paper is organized as follows; in Section 2, we build an exact formula for ruin (non-ruin)
probability for model (1.1) and (1.2) with \( X = \{X_{t}\}_{t \geq 0} \) and \( Y = \{Y_{j}\}_{j \geq 1} \) are independent and
identically distributed positive-valued random variables; \( I = \{I_{k}\}_{k \geq 1} \) are independent
identically distributed non-negative-valued random variables, \( X, Y \) and \( I \) are assumed to be
independent. An extended result in Section 2 for \( X, Y \) and \( I \) being non identically distributed
random variables is given in Section 3. A numerical example is given to illustrate these results
in Section 4. Finally, we conclude our paper in Section 5.
2. Finite – Time Ruin Probability in a Generalized Risk Processes under Interest Force with sequences of independent and identically distributed random variables

Let model (1.1). We assume that:

**Assumption 2.1.** $u$, $t$ are positive integer numbers.

**Assumption 2.2.** $X = \{X_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables, $X_n$ take values in a finite set of positive numbers

$$E_X = \{x_1, x_2, ..., x_M\} (0 < x_1 < x_2 < ... < x_M) \text{ with } p_k = P(X_1 = x_k) (x_k \in E_X), \quad 0 \leq p_k \leq 1, \sum_{k=1}^{M} p_k = 1.$$

**Assumption 2.3.** $Y = \{Y_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables, $Y_n$ take values in a finite set of positive numbers

$$E_Y = \{y_1, y_2, ..., y_N\} (0 < y_1 < y_2 < ... < y_N) \text{ with } q_k = P(Y_1 = y_k) (y_k \in E_Y), \quad 0 \leq q_k \leq 1, \sum_{k=1}^{N} q_k = 1.$$

**Assumption 2.4.** $I = \{I_n\}_{n \geq 1}$ is a sequence of independent and identically distributed random variables, $I_n$ take values in a finite set of non-negative numbers

$$E_I = \{i_1, i_2, ..., i_R\} (0 \leq i_1 < i_2 < ... < i_R) \text{ with } r_k = P(I_1 = i_k) (i_k \in E_I), \quad 0 \leq r_k \leq 1, \sum_{k=1}^{R} r_k = 1.$$

**Assumption 2.5.** The sequences $\{X_n\}_{n \geq 1}$, $\{Y_n\}_{n \geq 1}$, and $\{I_n\}_{n \geq 1}$ are assumed to be independent.

From (1.1), we have:

$$U_t = u \sum_{k=1}^{t} (1 + I_k) + \sum_{k=1}^{t} \left( X_k - Y_k \right) \prod_{j=k+1}^{t} (1 + I_j) + X_t - Y_t.$$  \hspace{1cm} (2.1)

where throughout this paper, we denote $\prod_{i=a}^{b} x_i = 1$ and $\sum_{i=a}^{b} x_i = 0$ if $a > b$

and $A \uparrow B$ if $P(A \Delta B) = 0$ with $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Supposing that the ruin time is defined by $T_u = \inf\{t : U_t < 0\}$, where $\inf \phi = \infty$.

We define the finite time ruin (non-ruin) probabilities of model (1.1) with assumption 2.1 to assumption 2.4, respectively, by

$$\psi_t^{(1)}(u) = P(T_u \leq t) = P\left( \bigcup_{j=1}^{t} (U_j < 0) \right),$$  \hspace{1cm} (2.2)
\[ \phi_{i}^{(1)}(u) = I - \psi_{i}^{(1)}(u) = P(T_u \geq t + 1) = P\left( \bigcap_{j=1}^{t}(U_j \geq 0) \right). \]  \hfill (2.3)

To establish a formula for \( \psi_{i}^{(1)}(u), \phi_{i}^{(1)}(u) \), we first prove the following Lemma.

**Lemma 2.1.** Let \( u, \{x_{ij}\}_{i,j=1}^{u}, \{y_{ij}\}_{i,j=1}^{u} \) be positive numbers, \( \{t_k\}_{k=1}^{t} \) be non-negative numbers.

If \( p \) is a positive integer number and \( 1 \leq p \leq t - 1 \) satisfies:

\[ y_p \leq u \prod_{k=1}^{p}(1+i_k) + \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p}(1+i_j) + x_p, \]  \hfill (2.4)

then, we have

\[ u \prod_{k=1}^{p+1}(1+i_k) + \sum_{k=1}^{p}(x_k - y_k) \prod_{j=k+1}^{p+1}(1+i_j) + x_{p+1} > 0. \]  \hfill (2.5)

**Proof.**

From (2.4), we have

\[ y_p \leq u \prod_{k=1}^{p}(1+i_k) + \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p}(1+i_j) + x_p. \]

The above inequality is equivalent to

\[ x_p - y_p \geq -u \prod_{k=1}^{p}(1+i_k) - \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p}(1+i_j). \]

This inequality implies that

\[
\begin{align*}
& u \prod_{k=1}^{p+1}(1+i_k) + \sum_{k=1}^{p}(x_k - y_k) \prod_{j=k+1}^{p+1}(1+i_j) + x_{p+1} \\
& = u \prod_{k=1}^{p+1}(1+i_k) + \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p+1}(1+i_j) + (x_p - y_p)(1+i_{p+1}) + x_{p+1} \\
& \geq u \prod_{k=1}^{p+1}(1+i_k) + \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p+1}(1+i_j) + \left[ -u \prod_{k=1}^{p}(1+i_k) - \sum_{k=1}^{p-1}(x_k - y_k) \prod_{j=k+1}^{p}(1+i_j) \right] (1+i_{p+1}) + x_{p+1} \\
& = x_{p+1} > 0.
\end{align*}
\]

Hence (2.5) holds.

This completes the proof of the Lemma 2.1. \( \square \)

Now, we give an exact formula for finite time ruin (non-ruin) probability of model (1.1).

**Theorem 2.1.** If model (1.1) satisfies assumptions 2.1 to 2.5, then finite time non-ruin probability of model (1.1) is defined by
\[ \varphi_t^{(1)}(u) = \sum_{c_1, c_2, \ldots, c_t = 1}^B \sum_{m_1, m_2, \ldots, m_t = 1}^M r_{c_1} r_{c_2} \cdots r_{c_t} p_{m_1} p_{m_2} \cdots p_{m_t} \left( \sum_{1 \leq n_1 \leq g_1} \sum_{1 \leq n_2 \leq g_2} \cdots \sum_{1 \leq n_t \leq g_t} q_{n_1} q_{n_2} \cdots q_{n_t} \right), \] (2.6)

where

\[
\begin{align*}
& g_1 = \max \left\{ n_1 : y_{n_1} \leq \min \left\{ u \prod_{k=1}^1 (1 + i_{c_k}) + x_{m_1}, y_N \right\} \right\}, \\
& g_2 = \max \left\{ n_2 : y_{n_2} \leq \min \left\{ u \prod_{k=1}^2 (1 + i_{c_k}) + \sum_{k=1}^1 (x_{m_k} - y_{n_k}) \prod_{j=k+1}^2 (1 + i_{c_j}) + x_{m_2}, y_N \right\} \right\}, \\
& \quad \vdots \\
& g_t = \max \left\{ n_t : y_{n_t} \leq \min \left\{ u \prod_{k=1}^t (1 + i_{c_k}) + \sum_{k=1}^{t-1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^t (1 + i_{c_j}) + x_{m_t}, y_N \right\} \right\}.
\end{align*}
\]

**Proof.**

Firstly, we have

\[
A := \bigcap_{j=1}^t (U_j \geq 0)
\]

\[
= \left\{ Y_j \leq u \prod_{k=1}^1 (1 + I_k) + X_1 \right\} \cap \left\{ Y_2 \leq u \prod_{k=1}^2 (1 + I_k) + \sum_{k=1}^1 (X_k - Y_k) \prod_{j=k+1}^2 (1 + I_j) + X_2 \right\} \cap \left\{ Y_3 \leq u \prod_{k=1}^3 (1 + I_k) + \sum_{k=1}^2 (X_k - Y_k) \prod_{j=k+1}^3 (1 + I_j) + X_3 \right\} \cap \cdots
\]

\[
\cdots \cap \left\{ Y_t \leq u \prod_{k=1}^t (1 + I_k) + \sum_{k=1}^{t-1} (X_k - Y_k) \prod_{j=k+1}^t (1 + I_j) + X_t \right\}.
\] (2.7)

By assumption 2.4, we put \( I_1 = i_{c_1}, I_2 = i_{c_2}, \ldots, I_t = i_{c_t} \) with \( i_{c_1}, i_{c_2}, \ldots, i_{c_t} \) being non-negative numbers and satisfy condition: \( 0 \leq i_{c_1}, i_{c_2}, \ldots, i_{c_t} \leq i_R \).

Let \( A_{i_{c_1}, \ldots, i_{c_t}} = (I_1 = i_{c_1}) \cap (I_2 = i_{c_2}) \cap \ldots \cap (I_t = i_{c_t}). \)

Since \( I = \{ I_n \}_{n \geq 1} \) is a sequence of independent random variables then

\[
P(A_{i_{c_1}, \ldots, i_{c_t}}) = P[(I_1 = i_{c_1}) \cap (I_2 = i_{c_2}) \cap \ldots \cap (I_t = i_{c_t})]
\]

\[
= P(I_1 = i_{c_1})P(I_2 = i_{c_2}) \cdots P(I_t = i_{c_t}) = r_{c_1} r_{c_2} \cdots r_{c_t}.
\] (2.8)

By Assumption 2.2, we put \( X_1 = x_{m_1}, X_2 = x_{m_2}, \ldots, X_t = x_{m_t} \) with \( x_{m_1}, x_{m_2}, \ldots, x_{m_t} \) being positive numbers and satisfy condition: \( 0 < x_{m_1}, x_{m_2}, \ldots, x_{m_t} \leq x_M \).
Let \( A_{x_1, x_2, \ldots, x_n} = (X_1 = x_1) \cap (X_2 = x_2) \cap \ldots \cap (X_n = x_n) \).

Since \( X = \{X_i\}_{n \in \mathbb{N}} \) is a sequence of independent random variables then
\[
P(A_{x_1, x_2, \ldots, x_n}) = P\left[ (X_1 = x_1) \cap (X_2 = x_2) \cap \ldots \cap (X_n = x_n) \right]
\]
\[
= P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \ldots \cdot P(X_n = x_n) = P_{x_1} P_{x_2} \ldots P_{x_n}.
\]

(2.9)

Firstly, we consider \( I_1 = i_{c_1} \) \((c_1 = 1, R)\) then (2.7) is given
\[
A = \bigcup_{c_1=1}^{R} \left( I_1 = i_{c_1} \right) \cap \left( Y_1 \leq u \prod_{c_1=1}^{1} (1 + i_{c_1}) + X_1 \right) \cap \cdots
\]
\[
\cdots \cap \left( Y_j \leq u \prod_{k=2}^{2} (1 + I_k) + \sum_{k=1}^{1} (X_k - Y_k) \prod_{j=k+1}^{2} (1 + I_j) + X_2 \right)
\]

Similarly, we consider \( I_2 = i_{c_2} \ldots, I_t = i_{c_t} \) \((c_2, \ldots, c_t = 1, R)\), (2.7) can be written as
\[
A = \bigcup_{c_1, c_2, \ldots, c_t=1}^{R} \left( \left( I_1 = i_{c_1} \right) \cap \left( I_2 = i_{c_2} \right) \cap \ldots \cap \left( I_t = i_{c_t} \right) \right) \cap \left( Y_1 \leq u \prod_{k=1}^{1} (1 + i_{c_k}) + X_1 \right) \cap \cdots
\]
\[
\cdots \cap \left( Y_t \leq u \prod_{k=2}^{t} (1 + I_k) + \sum_{k=1}^{t-1} (X_k - Y_k) \prod_{j=k+1}^{t} (1 + I_j) + X_t \right).
\]

Next, we consider \( X_j = x_{m_j} \) \((m_j = 1, M)\), then
\[
A = \bigcup_{c_1, c_2, \ldots, c_t=1}^{R} \left( \left( I_1 = i_{c_1} \right) \cap \left( I_2 = i_{c_2} \right) \cap \ldots \cap \left( I_t = i_{c_t} \right) \right) \cap \bigcup_{m_1=1}^{M} \left( X_1 = x_{m_1} \right) \cap \left( Y_1 \leq u \prod_{k=1}^{1} (1 + i_{c_k}) + x_{m_k} \right) \cap \cdots
\]
\[
\cdots \cap \left( Y_j \leq u \prod_{k=2}^{j} (1 + I_k) + \sum_{k=1}^{j-1} (x_{m_k} - Y_k) \prod_{j=k+1}^{j} (1 + I_j) + X_2 \right) \cap
\]
\[
\begin{aligned}
Y_j &\leq u\prod_{k=1}^3 (1+i_{k_j}) + \left( x_{m_1} - Y_j \right) + \sum_{k=2}^i (X_k - Y_j) \prod_{j=k+1}^3 (1+i_{j}) + X_j ) \cap \\
\cap \left( Y_j &\leq u\prod_{k=1}^3 (1+i_{k_j}) + \left( x_{m_1} - Y_j \right) + \sum_{k=2}^i (X_k - Y_j) \prod_{j=k+1}^3 (1+i_{j}) + X_j ) \right)
\end{aligned}
\]

Similarly, we consider \( X_2 = x_{m_2}, ..., X_j = x_{m_j} (m_2, ..., m_j = 1, M) \) (2.7) can be rearranged as

\[
A = \bigcup_{c_1, c_2, ..., c_j=0}^R \left\{ \left( I_1 = c_1 \right) \cap \left( I_2 = c_2 \right) \cap ... \cap \left( I_j = c_j \right) \right\} \cap \bigcup_{m_1, m_2, ..., m_j=1}^M \left\{ \left( X_1 = x_{m_1} \right) \cap \left( X_2 = x_{m_2} \right) \cap ... \cap \left( X_j = x_{m_j} \right) \right\} \cap C_{m_1 m_2 \cdots m_j}^{(c_1 c_2 \cdots c_j)}
\]

\[
= \bigcup_{c_1, c_2, ..., c_j=0}^R \left\{ \left( I_1 = c_1 \right) \cap \left( I_2 = c_2 \right) \cap ... \cap \left( I_j = c_j \right) \right\} \cap \bigcup_{m_1, m_2, ..., m_j=1}^M \left\{ \left( X_1 = x_{m_1} \right) \cap \left( X_2 = x_{m_2} \right) \cap ... \cap \left( X_j = x_{m_j} \right) \right\} \cap C_{m_1 m_2 \cdots m_j}^{(c_1 c_2 \cdots c_j)}
\]

(2.10)

where

\[
\begin{aligned}
C_{m_1 m_2 \cdots m_j}^{(c_1 c_2 \cdots c_j)} &\equiv Y_1 \leq u\prod_{k=1}^1 (1+i_{k_1}) + x_{m_1} \cap Y_2 \leq u\prod_{k=1}^2 (1+i_{k_2}) + \sum_{k=2}^i (x_{m_k} - Y_j) \prod_{j=k+1}^2 (1+i_{j}) + x_{m_2} \cap \\
\cap \left( Y_j &\leq u\prod_{k=1}^3 (1+i_{k_j}) + \left( x_{m_1} - Y_j \right) + \sum_{k=2}^i (X_k - Y_j) \prod_{j=k+1}^3 (1+i_{j}) + X_j ) \right) \right)
\end{aligned}
\]

(2.11)

By assumption 2.3, we put \( Y_1 = y_{n_1}, Y_2 = y_{n_2}, ..., Y_{n-1} = y_{n_{n-1}} \) with \( y_{n_1}, y_{n_2}, ..., y_{n_{n-1}} \) being positive numbers and satisfy condition: \( 0 < y_{n_1}, y_{n_2}, ..., y_{n_{n-1}} \leq y_N \).

Thus, (2.11) can be written as
\[ C_{\mathcal{X}^n_1 \cdots \mathcal{X}^n_m} \supseteq \bigcup_{y_n \in \mathcal{X}^n} \left( Y_1 = y_n \right) \cap \left( \begin{array}{l} Y_2 \leq u \prod_{k=1}^{2} (1+i_{c_k}) + \sum_{k=1}^{1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2} (1+i_{c_j}) + x_{m_2} \end{array} \right) \cap \\
\left( Y_3 \leq u \prod_{k=1}^{3} (1+i_{c_k}) + \sum_{k=2}^{2} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{3} (1+i_{c_j}) + x_{m_3} \right) \cap \ldots \cap \\
\left( Y_\ell \leq u \prod_{k=1}^{\ell} (1+i_{c_k}) + \sum_{k=\ell}^{\ell-1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{\ell} (1+i_{c_j}) + x_{m_\ell} \right) \right). \tag{2.12} \]

Using by assumption 2.3, we put \( Y_i = y_{n_i} \) with \( y_{n_i} \) being positive number and satisfy condition \( 0 < y_{n_i} \leq y_{n_i} \) then (2.11) can be rearranged as

\[ C_{\mathcal{X}^n_1 \cdots \mathcal{X}^n_m} \supseteq \bigcup_{y_n \in \mathcal{X}^n} \left( Y_1 = y_n \right) \cap \bigcup_{y_{n_2} \in \mathcal{X}^{n_2}} \left( \begin{array}{l} Y_2 = y_{n_2} \end{array} \right) \cap \left( Y_3 = y_{n_3} \right) \cap \ldots \cap \left( Y_\ell = y_{n_\ell} \right) \right). \tag{2.13} \]

By using Lemma 2.1, \( u \prod_{k=1}^{1} (1+i_{c_k}) + x_{m_1} \),
\[
\begin{align*}
&u \prod_{k=1}^{2} (1 + i_{c_k}) + \sum_{k=1}^{1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2} (1 + i_{c_j}) + x_{m_2}, \\
&u \prod_{k=1}^{t} (1 + i_{c_k}) + \sum_{k=1}^{t} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{t} (1 + i_{c_j}) + x_{m_t}
\end{align*}
\]

are positive numbers and

\[
0 < y_{n_1}, y_{n_2}, \ldots, y_{n_t} \leq y_N \text{ then, we define}
\]

\[
g_1 = \max \left\{ n_1 : y_{n_1} \leq \min \left\{ u \prod_{k=1}^{1} (1 + i_{c_k}) + x_{m_1}, y_N \right\} \right\},
\]

\[
g_2 = \max \left\{ n_2 : y_{n_2} \leq \min \left\{ u \prod_{k=1}^{2} (1 + i_{c_k}) + \sum_{k=1}^{1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{2} (1 + i_{c_j}) + x_{m_2}, y_N \right\} \right\},
\]

\[
\ldots
\]

\[
g_t = \max \left\{ n_t : y_{n_t} \leq \min \left\{ u \prod_{k=1}^{t} (1 + i_{c_k}) + \sum_{k=1}^{t-1} (x_{m_k} - y_{n_k}) \prod_{j=k+1}^{t} (1 + i_{c_j}) + x_{m_t}, y_N \right\} \right\}.
\]

Thus, (2.13) can be rearranged as

\[
C^{x_{m_1}, \ldots, x_{m_t}}_{y_{n_1}, \ldots, y_{n_t}} \equiv \bigcup_{1 \leq n_1 \leq g_1} \bigcup_{1 \leq n_2 \leq g_2} \ldots \bigcup_{1 \leq n_t \leq g_t} \left\{ (Y_1 = y_{n_1}) \cap (Y_2 = y_{n_2}) \cap \ldots \cap (Y_t = y_{n_t}) \right\}.
\]

(2.14)

Because \( Y = \{ Y_n \}_{n \geq 1} \) is a sequence of independent independent random variables then

\[
P\left[ (Y_1 = y_{n_1}) \cap (Y_2 = y_{n_2}) \cap \ldots \cap (Y_t = y_{n_t}) \right] = P(Y_1 = y_{n_1})P(Y_2 = y_{n_2}) \ldots P(Y_t = y_{n_t}) = q_{n_1} q_{n_2} \ldots q_{n_t}
\]

In the other hand, system of events \( \left\{ (Y_1 = y_{n_1}) \cap (Y_2 = y_{n_2}) \cap \ldots \cap (Y_t = y_{n_t}) \right\}_{1 \leq n_j \leq g_j (j = \{1, \ldots, t\})} \) in

(2.14) be incompatible then

\[
P(B_{x_{m_1}, \ldots, x_{m_t}}) = \sum_{1 \leq n_1 \leq g_1} \sum_{1 \leq n_2 \leq g_2} \ldots \sum_{1 \leq n_t \leq g_t} q_{n_1} q_{n_2} \ldots q_{n_t}.
\]

(2.15)

By \( X, Y, I \) are assumed to be independent, with \( c_1, c_2, \ldots, c_r \) and \( m_1, m_2, \ldots, m_t \) hold then

\( A_{c_1, \ldots, c_r}, B_{x_{m_1}, \ldots, x_{m_t}}, C_{l_{1}, \ldots, l_{r}} \) are independent events.

In addition, system of events \( \left\{ A_{c_1, \ldots, c_r} \cap B_{x_{m_1}, \ldots, x_{m_t}} \cap C_{l_{1}, \ldots, l_{r}} \right\} \) in (2.10) is

incompatible.

Therefore, combining (2.8), (2.9) and (2.15), we have
\[ \phi_t^{(1)}(u) = P(A) = \sum_{c_1, c_2, \ldots, c_l} \left( \sum_{m_1, m_2, \ldots, m_l} P(A_{c_1, m_2, \ldots, m_l} \cap B_{m_1, m_2, \ldots, m_l} \cap C_{m_1, m_2, \ldots, m_l}) \right) \]
\[ = \sum_{c_1, c_2, \ldots, c_l} \left( \sum_{m_1, m_2, \ldots, m_l} P(A_{c_1, m_2, \ldots, m_l}) \cdot P(B_{m_1, m_2, \ldots, m_l}) \cdot P(C_{m_1, m_2, \ldots, m_l}) \right) \]
\[ = \sum_{c_1, c_2, \ldots, c_l} \left( \sum_{m_1, m_2, \ldots, m_l} \prod_{k=1}^l r_{c_k} \prod_{j=1}^l P(m_j) \right) \cdot \sum_{i=1}^l \sum_{j=1}^l \sum_{k=1}^l q_i q_j q_k \ldots q_l. \]  

This completes the proof of the Theorem 2.1. \( \square \)

**Corollary 2.1.** If model (1.1) satisfies assumptions 2.1 to 2.4, then finite time ruin probability of model (1.1) is defined by

\[ \psi_t^{(1)}(u) = 1 - \phi_t^{(1)}(u) \]
\[ = 1 - \sum_{c_1, c_2, \ldots, c_l} \left( \sum_{m_1, m_2, \ldots, m_l} \prod_{k=1}^l r_{c_k} \prod_{j=1}^l P(m_j) \right) \cdot \sum_{i=1}^l \sum_{j=1}^l \sum_{k=1}^l q_i q_j q_k \ldots q_l. \]  

**Remark 2.1.** Formula (2.6) (or (2.17)) gives a method to compute exactly finite time non-ruin (ruin) probability of model (1.1) which \( X = \{X_n\}_{n \geq 1} \) and \( Y = \{Y_n\}_{n \geq 1} \) are sequences of independent and identically distributed random variables, they take values in a finite set of positive numbers and \( I = \{I_n\}_{n \geq 1} \) is a sequence of independent and identically distributed random variables, and they take values in a finite set of non-negative numbers.

Let model (1.2) satisfy assumptions 2.1 to 2.5.

From (1.2), we have:

\[ U_t = u \cdot \prod_{k=1}^t (1 + I_k) + \sum_{k=1}^t \left( X_k (1 + I_k) - Y_k \right) \prod_{j=k+1}^t (1 + I_j) + X_t - Y_t. \]  

Supposing that the ruin time of model (1.2) is defined by \( T_\phi = \inf \{ j : U_j < 0 \} \), where \( \inf \phi = \infty \).

We define the finite time ruin (non-ruin) probabilities of model (1.2) with assumptions 2.1 to 2.5, respectively, by

\[ \psi_t^{(2)}(u) = P(T_u \leq t) = P(\bigcup_{k=1}^t (U_k < 0)), \]  

(2.19)
\[
\phi_t^{(2)}(u) = 1 - \psi_t^{(2)}(u) = P(T_u \geq t + 1) = P \left( \bigcap_{k=1}^{t} (U_k \geq 0) \right). \tag{2.20}
\]

To establish an formula for \( \psi_t^{(2)}(u), \phi_t^{(2)}(u) \), we have the following Lemma.

**Lemma 2.2.** Let \( u, \{y_{ij}\}_{i,j=1}^{l}, \{y_{ij}\}_{i,j=1}^{l} \) are positive numbers and \( \{i_{k}\}_{k=1}^{l} \) are non-negative numbers.

If \( p \) is a positive integer number and \( 1 \leq p \leq t-1 \) satisfies:

\[
y_p \leq u \prod_{k=1}^{p-1} (1 + i_k) + \sum_{k=1}^{p} (x_k (1 + i_k) - y_k) \prod_{j=k+1}^{p} (1 + i_j) + x_p (1 + i_p), \tag{2.21}
\]

then, we have

\[
u \prod_{k=1}^{p+1} (1 + i_k) + \sum_{k=1}^{p} (x_k (1 + i_k) - y_k) \prod_{j=k+1}^{p} (1 + i_j) + x_{p+1} (1 + i_{p+1}) > 0. \tag{2.22}
\]

**Proof.**

We proof similarly as Lemma 2.1. \( \Box \)

Next, we give an exact formula for finite time ruin (non ruin) probability of model (1.1).

**Theorem 2.2.** If model (1.2) satisfies assumptions 2.1 to 2.5, then finite time non-ruin probability of model (1.2) is defined by

\[
\phi_t^{(2)}(u) = \sum_{c_1,c_2,...,c_t=1}^{R} \sum_{m_1,m_2,...,m_t=1}^{M} r_{c_1}r_{c_2}...r_{c_t} p_{m_1}p_{m_2}...p_{m_t} \left( \sum_{i_1,i_2,...,i_t=1}^{\infty} \sum_{s_1,s_2,...,s_t=1}^{\infty} \sum_{l_1,l_2,...,l_t=1}^{\infty} q_{i_1}q_{i_2}...q_{i_t} \right), \tag{2.23}
\]

where

\[
g_1 = \max \left\{ n_1 : y_{n_1} \leq \min \left\{ u \prod_{k=1}^{1} (1 + i_k) + x_{m_1} (1 + i_{c_1}), y_N \right\} \right\},
\]

\[
g_2 = \max \left\{ n_2 : y_{n_2} \leq \min \left\{ u \prod_{k=1}^{2} (1 + i_k) + \sum_{k=1}^{1} (x_{m_1} (1 + i_{c_1}) - y_{n_1}) \prod_{j=k+1}^{2} (1 + i_{c_j}) + x_{m_2} (1 + i_{c_2}), y_N \right\} \right\},
\]

... \[
\]

\[
g_t = \max \left\{ n_t : y_{n_t} \leq \min \left\{ u \prod_{k=1}^{t} (1 + i_k) + \sum_{k=1}^{t-1} (x_{m_1} (1 + i_{c_1}) - y_{n_1}) \prod_{j=k+1}^{t} (1 + i_{c_j}) + x_{m_t} (1 + i_{c_t}), y_N \right\} \right\}.
\]

**Proof.**

We proof similarly as Theorem 2.1. \( \Box \)

**Corollary 2.2.** If model (1.2) satisfies assumptions 2.1 to 2.5, then finite time ruin probability of model (1.2) is defined by

\[
\psi_t^{(2)}(u) = 1 - \phi_t^{(2)}(u)
\]
\begin{align}
1 - \sum_{c_1, c_2, \ldots, c_r = 1}^{R} \sum_{m_1, m_2, \ldots, m_r = 1}^{M} (r_1 c_1 \cdots r_r) \cdot (p_{m_1} p_{m_2} \cdots p_{m_r}) \left( \sum_{l_1 \in S_1} \sum_{l_2 \in S_2} \cdots \sum_{l_R \in S_R} q_{l_1} q_{l_2} \cdots q_{l_R} \right). \tag{2.24}
\end{align}

Remark 2.2. Formula (2.23) (or (2.24)) give a method to compute exact finite time non-ruin (ruin) probability of model (1.2) which $X = \{X_n\}_{n \geq 1}$ and $Y = \{Y_n\}_{n \geq 1}$ are sequences of independent and identically distributed random variables and they take values in a finite set of positive numbers. In addition, $I = \{I_n\}_{n \geq 1}$ is also a sequence of independent and identically distributed random variables, and they take values in a finite set of non-negative numbers.

3. Finite – Time Ruin Probability in a Generalized Risk Processes under Interest Force with sequences of independent and non identically distributed random variables

Let model (1.1). We assume that:

Assumption 3.1. $u$, $t$ are positive integer numbers.

Assumption 3.2. $X = \{X_n\}_{n \geq 1}$ is a sequence of independent and non identically distributed random variables, $X_n$ takes values in a finite set of positive numbers $E_X = \{x_1, x_2, \ldots, x_M\} (0 < x_1 < x_2 < \ldots < x_M)$ and $X_n$ has a distribution:

$$
p_{k}^{(n)} = P(X_n = x_k) (x_k \in E_X, n \in N^*), \quad 0 \leq p_{k}^{(n)} \leq 1, \sum_{k=1}^{M} p_{k}^{(n)} = 1 (n \in N^*).
$$

Assumption 3.3. $Y = \{Y_n\}_{n \geq 1}$ is a sequence of independent and non identically distributed random variables, $Y_n$ takes values in a finite set of positive integer numbers $E_Y = \{y_1, y_2, \ldots, y_N\} (0 < y_1 < y_2 < \ldots < y_N)$ and $Y_n$ has a distribution:

$$
q_{k}^{(n)} = P(Y_n = y_k) (y_k \in E_Y, n \in N^*), \quad 0 \leq q_{k}^{(n)} \leq 1, \sum_{k=1}^{N} q_{k}^{(n)} = 1 (n \in N^*).
$$

Assumption 3.4. $I = \{I_n\}_{n \geq 1}$ is a sequence of independent and non identically distributed random variables, $I_n$ takes values in a finite set of non-negative numbers $E_I = \{i_1, i_2, \ldots, i_R\} (0 \leq i_1 < i_2 < \ldots < i_R)$ and $I_n$ has a distribution:

$$
r_{k}^{(n)} = P(I_n = r_k) (r_k \in E_I, n \in N^*), \quad 0 \leq r_{k}^{(n)} \leq 1, \sum_{k=1}^{R} r_{k}^{(n)} = 1 (n \in N^*).
$$

Assumption 3.5. The sequences $\{X_n\}_{n \geq 1}$, $\{Y_n\}_{n \geq 1}$, and $\{I_n\}_{n \geq 1}$ are assumed to be independent.
Supposing that the ruin time of model (1.1) is defined by \( T_u = \inf \{ j : U_j < 0 \} \) where \( \inf \phi = \infty \).

We define the finite time ruin (non-ruin) probabilities of model (1.1) with assumptions 3.1 to 3.5, respectively, by

\[
\psi_r^{(3)}(u) = P(T_u \leq t) = P \left( \bigcup_{k=1}^{t} (U_k < 0) \right),
\]

\[
\phi_r^{(3)}(u) = 1 - \psi_r^{(3)}(u) = P(T_u \geq t + 1) = P \left( \bigcap_{k=1}^{t} (U_k \geq 0) \right).
\]

Similar to Theorem 2.1, we have

**Theorem 3.1.** If model (1.1) satisfies assumptions 3.1 to 3.5, then finite time non-ruin probability of model (1.1) is defined by

\[
\phi_r^{(3)}(u) = \sum_{c_1,c_2,\ldots,c_t} \sum_{m_1,m_2,\ldots,m_t=1}^{R} r_{c_1}^{(t)} r_{c_2}^{(t)} \ldots r_{c_t}^{(t)} p_{m_1}^{(t)} p_{m_2}^{(t)} \ldots p_{m_t}^{(t)} \left( \sum_{l=1}^{t_{1}} \sum_{l=1}^{t_{2}} \ldots \sum_{l=1}^{t_{n}} q_{l_n}^{(2)} q_{l_n}^{(2)} \ldots q_{l_n}^{(2)} \right),
\]

where, \( \xi_a, \xi_b, \ldots, \xi_c \) is defined in the same way with Theorem 2.1.

**Proof.**

We proof similarly as Theorem 2.1, where

(2.8) substituted by

\[
P(A_{i_1,i_2,\ldots,i_t}) = P \left[ (I_1 = i_{c_1}) \cap (I_2 = i_{c_2}) \cap \ldots \cap (I_t = i_{c_t}) \right]
\]

\[
= P(I_1 = i_{c_1}) P(I_2 = i_{c_2}) \ldots P(I_t = i_{c_t}) = r_{c_1}^{(t)} r_{c_2}^{(t)} \ldots r_{c_t}^{(t)}.
\]

In addition (2.9) replaced by

\[
P(B_{x_{m_1},x_{m_2},\ldots,x_{m_t}}) = P \left[ (X_1 = x_{m_1}) \cap (X_2 = x_{m_2}) \cap \ldots \cap (X_t = x_{m_t}) \right]
\]

\[
= P(X_1 = x_{m_1}) P(X_2 = x_{m_2}) \ldots P(X_t = x_{m_t}) = p_{m_1}^{(t)} p_{m_2}^{(t)} \ldots p_{m_t}^{(t)},
\]

and (2.15) substituted by

\[
P(C_{x_{m_1},x_{m_2},\ldots,x_{m_t}}) = \sum_{l=1}^{t_{1}} \sum_{l=1}^{t_{2}} \ldots \sum_{l=1}^{t_{n}} q_{l_n}^{(2)} q_{l_n}^{(2)} \ldots q_{l_n}^{(2)}.
\]

By using the same method to prove Theorem 2.1, we have formula (3.3).

This completes the proof of the Theorem 3.1. \( \square \)

**Corollary 3.1.** If model (1.1) satisfies assumptions 3.1 to 3.5, then finite time ruin probability of model (1.1) is defined by

\[
\psi_r^{(3)}(u) = 1 - \phi_r^{(3)}(u)
\]
= 1 − \sum_{c_1, c_2, \ldots, c_l = 1}^{R} \sum_{m_1, m_2, \ldots, m_l = 1}^{M} \prod_{i=1}^{l} r_{c_i}^{(1)} r_{c_i}^{(2)} \cdots r_{c_i}^{(t)} p_{m_1}^{(1)} p_{m_2}^{(2)} \cdots p_{m_l}^{(t)} \left( \sum_{\mathcal{I}} \sum_{\mathcal{I}_2} \cdots \sum_{\mathcal{I}_n} q_{n_1}^{(1)} q_{n_2}^{(2)} \cdots q_{n_t}^{(t)} \right),\quad (3.4)

Remark 3.1. Formula (3.3) (or (3.4)) gives a method to compute exactly finite time non-ruin (ruin) probability of model (1.1) which \( X = \{X_n\}_{n \in \mathbb{N}} \) and \( Y = \{Y_n\}_{n \geq 1} \) are sequences of independent and non identically distributed random variables, they take values in a finite set of positive numbers. In addition, \( I = \{I_n\}_{n \geq 1} \) is a sequence of independent and non identically distributed random variables, and they take values in a finite set of non-negative numbers.

Similarly, we consider model (1.2) satisfy assumptions 3.1 to 3.5.

Supposing that the ruin time of model (1.2) is defined by \( T_u = \inf\{j : U_j < 0\} \) where \( \inf \phi = \infty \).

We define the finite time ruin (non-ruin) probabilities of model (1.2) with assumptions 3.1 to 3.5, respectively, by

\[
\psi_t^{(4)}(u) = P(T_u \leq t) = P\left( \bigcap_{k=1}^{t} (U_k < 0) \right),
\]

\[
\phi_t^{(4)}(u) = 1 - \psi_t^{(4)}(u) = P(T_u > t + 1) = P\left( \bigcup_{k=1}^{t} (U_k \geq 0) \right).
\]

Similar to Theorem 2.2, we have

**Theorem 3.2.** If model (1.2) satisfies assumptions 3.1 to 3.5, then finite time non-ruin probability of model (1.2) is defined by

\[
\phi_t^{(4)}(u) = \sum_{c_1, c_2, \ldots, c_l = 1}^{R} \sum_{m_1, m_2, \ldots, m_l = 1}^{M} \prod_{i=1}^{l} r_{c_i}^{(1)} r_{c_i}^{(2)} \cdots r_{c_i}^{(l)} (p_{m_1}^{(1)} p_{m_2}^{(2)} \cdots p_{m_l}^{(t)}) \left( \sum_{\mathcal{I}} \sum_{\mathcal{I}_2} \cdots \sum_{\mathcal{I}_n} q_{n_1}^{(1)} q_{n_2}^{(2)} \cdots q_{n_t}^{(t)} \right),
\]

where, \( g_1, g_2, \ldots, g_t \) is defined in the same way with Theorem 2.2.

**Proof.**

We proof similarly as Theorem 3.1. \( \square \)

**Corollary 3.2.** If model (1.2) satisfies assumptions 3.1 to 3.5, then finite time ruin probability of model (1.2) is defined by

\[
\psi_t^{(4)}(u) = 1 - \phi_t^{(4)}(u)
\]

\[
= 1 - \sum_{c_1, c_2, \ldots, c_l = 1}^{R} \sum_{m_1, m_2, \ldots, m_l = 1}^{M} \prod_{i=1}^{l} r_{c_i}^{(1)} r_{c_i}^{(2)} \cdots r_{c_i}^{(l)} (p_{m_1}^{(1)} p_{m_2}^{(2)} \cdots p_{m_l}^{(t)}) \left( \sum_{\mathcal{I}} \sum_{\mathcal{I}_2} \cdots \sum_{\mathcal{I}_n} q_{n_1}^{(1)} q_{n_2}^{(2)} \cdots q_{n_t}^{(t)} \right).
\]

(3.8)
Remark 3.2. Formula (3.7) (or (3.8)) gives a method to compute exactly finite time non-ruin (ruin) probability of model (1.2) which \( X = \{X_n\}_{n \geq 1} \) and \( Y = \{Y_n\}_{n \geq 1} \) are sequences of independent and non identically distributed random variables, they take values in a finite set of positive numbers. In addition, \( I = \{I_n\}_{n \geq 1} \) is a sequence of independent and non identically distributed random variables, and they take values in a finite set of non-negative numbers.

4. A numerical Illustration

4.1. A numerical Illustration for \( \psi_{t}^{(1)}(u) \)

Let \( X = \{X_n\}_{n \geq 1} \) be a sequence of independent and identically distributed random variables, \( X_n \) takes values in a finite set of positive integer numbers \( E_X = \{1,2,3,4\} \) with \( X \) having a distribution:

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0,475112</td>
<td>0,176783</td>
<td>0,153448</td>
<td>0,194657</td>
</tr>
</tbody>
</table>

Let \( Y = \{Y_n\}_{n \geq 1} \) be a sequence of independent and identically distributed random variables, \( Y_n \) take values in a finite set of positive integer numbers \( E_Y = \{1,2,3,4\} \) with \( Y_1 \) having a distribution:

<table>
<thead>
<tr>
<th>( Y_1 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0,910703</td>
<td>0,009639</td>
<td>0,026892</td>
<td>0,052766</td>
</tr>
</tbody>
</table>

Let \( I = \{I_n\}_{n \geq 1} \) be a sequence of independent and identically distributed random variables, \( I_n \) take values in a finite set of positive integer numbers \( E_I = \{0,1;0,11;0,12;0,13\} \) with \( I_1 \) having a distribution:

<table>
<thead>
<tr>
<th>( I_1 )</th>
<th>0,10</th>
<th>0,11</th>
<th>0,12</th>
<th>0,13</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0,758171</td>
<td>0,228950</td>
<td>0,002498</td>
<td>0,010380</td>
</tr>
</tbody>
</table>

By using the C program, the \( \psi_{t}^{(1)}(u) \) is calculated with the assumptions above of random variables \( X_1, Y_1, I_1 \). Table 4.1 shows \( \psi_{t}^{(1)}(u) \) for a range of value of \( u \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( t = 3 )</td>
</tr>
</tbody>
</table>
4.2. A numerical Illustration for $\psi^{(2)}_r(u)$

Let $X = \{X_n\}_{n \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables, $X_n$ take values in a finite set of positive integer numbers $E_X = \{1, 2, 3, 4\}$ with $X_i$ having a distribution:

<table>
<thead>
<tr>
<th>$X_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.910367</td>
<td>0.042479</td>
<td>0.045050</td>
<td>0.002104</td>
</tr>
</tbody>
</table>

Let $Y = \{Y_n\}_{n \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables, $Y_n$ take values in a finite set of positive integer numbers $E_Y = \{1, 2, 3, 4\}$ with $Y_i$ having a distribution:

<table>
<thead>
<tr>
<th>$Y_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.326243</td>
<td>0.184154</td>
<td>0.115890</td>
<td>0.373713</td>
</tr>
</tbody>
</table>

Let $I = \{I_n\}_{n \in \mathbb{N}}$ be a sequence of independent and identically distributed random variables, $I_n$ take values in a finite set of positive integer numbers $E_I = \{0, 1; 0, 11; 0, 12; 0, 13\}$ with $I_i$ having a distribution:

<table>
<thead>
<tr>
<th>$I_i$</th>
<th>0.1</th>
<th>0.11</th>
<th>0.12</th>
<th>0.13</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.481185</td>
<td>0.103107</td>
<td>0.261119</td>
<td>0.154588</td>
</tr>
</tbody>
</table>

By using the C program, the $\psi^{(2)}_r(u)$ is calculated with the assumptions above of random variables $X, Y, I$.

Table 4.2 shows $\psi^{(2)}_r(u)$ for a range of value of $u$.

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$X_i$ & 1 & 2 & 3 \\
\hline
P & 0.910367 & 0.042479 & 0.045050 \\
\hline
\end{tabular}
\caption{Table 4.1. Ruin probabilities (1.1) with Assumption 2.1- Assumption 2.5.}
\end{table}
Table 4.2. Ruin probabilities (1.2) with Assumption 2.1- Assumption 2.5.

<table>
<thead>
<tr>
<th>u</th>
<th>t = 3</th>
<th>t = 4</th>
<th>t = 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>0.293167</td>
<td>0.327225</td>
<td>0.352079</td>
</tr>
<tr>
<td>2.5</td>
<td>0.155001</td>
<td>0.188188</td>
<td>0.213372</td>
</tr>
<tr>
<td>3.5</td>
<td>0.070132</td>
<td>0.097067</td>
<td>0.118840</td>
</tr>
<tr>
<td>4.5</td>
<td>0.032686</td>
<td>0.050891</td>
<td>0.067123</td>
</tr>
<tr>
<td>5.5</td>
<td>0.011821</td>
<td>0.023018</td>
<td>0.034128</td>
</tr>
<tr>
<td>6.5</td>
<td>0.003710</td>
<td>0.009619</td>
<td>0.016400</td>
</tr>
<tr>
<td>7.5</td>
<td>0.000996</td>
<td>0.003650</td>
<td>0.007374</td>
</tr>
</tbody>
</table>

5. Conclusion

Using technique of classical probability with u, t, claims, premiums which all are positive numbers and interests are non – negative numbers, this paper constructed an exact formula for ruin (non-ruin) probability for model (1.1) and model (1.2) where sequences of claims, premiums and interests are independent (non) identically distributed random variables. Our main results in this paper are not only Theorem 2.1, Theorem 2.2, Theorem 3.1 and Theorem 3.2. In addition, numerical examples are given to illustrate for Theorem 2.1 and Theorem 2.2. These results proof for the suitability of theoretical result and practical examples. It also means that:

- When initial u is increasing then \( \psi_i^{(1)}(u), \psi_i^{(2)}(u) \) are decreasing,
- With u being unchanged, when t is increasing then \( \psi_i^{(1)}(u), \psi_i^{(2)}(u) \) are increasing.

REFERENCES


Received: March, 2014