Energy decay of a thermoelastic system with nonlinear feedback

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Abstract

Using the multiplier method and the abstract setting from [7], we derive different stability results for an isotropic thermoelastic system with combined nonlinear internal and boundary feedbacks.

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1 Introduction

Let Ω be a non empty bounded open subset of $\mathbb{R}^n, n \geq 1$, with a boundary $\Gamma$ of class $C^2$. We denote by $\nu = (\nu_1, \cdots, \nu_n)$ the unit outward normal vector along $\Gamma$. For a fixed $x_0 \in \mathbb{R}^n$ we define the function $m(x) = x - x_0$, $x \in \mathbb{R}^n$ and the following partition of the boundary $\Gamma$:

\[ \Gamma_1 = \{ x \in \Gamma : m(x) \cdot \nu(x) \leq 0 \}, \]
\[ \Gamma_2 = \{ x \in \Gamma : m(x) \cdot \nu(x) > 0 \}. \]
In this paper we consider the system of isotropic thermoelasticity:

\[
\begin{cases}
  u'' - \mu \Delta u - (\lambda + \mu)\nabla \text{div} u + \alpha \nabla \theta + f(u') = 0 \quad \text{in } Q := \Omega \times \mathbb{R}^+,
  
  \theta' - \Delta \theta + \beta \text{div} u' = 0 \quad \text{in } Q,
  
  \theta = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+,
  
  u(\cdot, 0) = u_0, \quad u'(\cdot, 0) = u_1 \quad \text{in } \Omega,
\end{cases}
\]

\[u(\cdot, 0) = u_0, \quad u'(\cdot, 0) = u_1 \quad \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega, \tag{3}\]

where \(u = u(x, t) = (u_1(x, t), \ldots, u_n(x, t))\) denotes the displacement vector field, \(\theta = \theta(x, t)\) the temperature.

The function \(a\) is non-negative and belongs to \(C^1(\Gamma_2)\); the functions \(f(u) = (f_1(u), \ldots, f_n(u))\) and \(g(u) = (g_1(u), \ldots, g_n(u))\) are continuous and satisfy

\[
\begin{align*}
  f(0) &= g(0) = 0, \quad \tag{4}
  
  (f(x) - f(y)) \cdot (x - y) &\geq 0, \quad \forall x, y \in \mathbb{R}^n, \quad \tag{5}
  
  (g(x) - g(y)) \cdot (x - y) &\geq 0, \quad \forall x, y \in \mathbb{R}^n. \quad \tag{6}
\end{align*}
\]

The coupling parameters \(\alpha\) and \(\beta\) are supposed to be positive. Theses assumptions guarantee that the system (3) is dissipative since its energy defined by

\[
E(t) = \frac{1}{2} \int_{\Omega} \left\{ |u'|^2 + \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 + \frac{\alpha}{\beta} |\theta|^2 \right\} dx + \frac{1}{2} \int_{\Gamma_2} am \cdot \nu |u|^2 d\Gamma \quad \tag{7}
\]

is nonincreasing.

The stabilization of different variant of the system (3) has been studied in the literature, notably in [2, 4, 5, 6, 8, 10, 11] (see also [3] in the anisotropic case). In [4], Liu considered the case \(f = 0\) in the linear feedback, i.e., \(g(x) = x\) on \(\Gamma_2 \neq \emptyset\) and give exponential decay of energy. Still in the case \(f = 0\) Liu and Zuazua [5] have established exponential, polynomial and logarithmic decay for some nonlinearities \(g\).

The aims of this work is to generalize these results to the case \(f \neq 0\). For this purpose, in the linear case we establish integral inequalities as in [4] leading to the exponential decay and in the nonlinear case, we use the theorical results established in [7].

## 2 Main Results

In the remainder of our paper we suppose that

\[
\Gamma_1 \neq \emptyset \text{ or } a(x) > 0, \forall x \in \Gamma_2. \tag{8}
\]
Furthermore, in order to avoid regularity problems related to the change of boundary conditions we assume that
\[ \Gamma_1 \cap \Gamma_2 = \emptyset. \] (9)

We finally suppose that there exist positive constants \( C > 0 \) and \( \sigma \geq 0, \sigma' \geq 0 \) such that
\[
|f(x)| \leq \begin{cases} C[1 + |x|^{n+2}] & \text{if } n \geq 3, \\ C[1 + |x|^n] & \text{if } n \leq 2, \end{cases} \tag{10}
\]
\[
|g(x)| \leq \begin{cases} C[1 + |x|^{n+2}] & \text{if } n \geq 3, \\ C[1 + |x|^n] & \text{if } n \leq 2 \end{cases} \tag{11}
\]

We define the following Hilbert spaces:
\[
H^1_{\Gamma_1}(\Omega) = \{ u \in H^1(\Omega); u = 0 \text{ on } \Gamma_1 \},
\]
\[
D_{\Gamma_1} = \{ (u, v, \theta) \in (H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega))^n \times (H^1_{\Gamma_1}(\Omega))^n \times (H^2(\Omega) \cap H^1_0(\Omega)) : \mu \partial_n u + (\lambda + \mu) \text{div } u \nu + am \cdot \nu u + m \cdot \nu v = 0 \text{ on } \Gamma_2 \},
\]
\[
W = (H^1_{\Gamma_1}(\Omega))^n \times (L^2(\Omega))^n,
\]
\[
H = W \times L^2(\Omega).
\]

The space \( W \) is equipped with the natural norm:
\[
\|(u, v)\|_W = \int_{\Omega} [v]^2 + \mu |\nabla u|^2 + (\lambda + \mu)|\text{div } u|^2 dx + \int_{\Gamma_2} am \cdot \nu |u|^2 d\Gamma.
\]

In the sequel, we denote by \( < \cdot, \cdot > \) the duality pairing between \((H^1_{\Gamma_1}(\Omega))^n\) and \([((H^1_{\Gamma_1}(\Omega))^n)^{\prime}\) or between \(H^1_0(\Omega)\) and \(H^{-1}(\Omega)\), and by \((\cdot, \cdot)\) the inner product in \((H^1_{\Gamma_1}(\Omega))^n\).

**Theorem 2.1** Let \( \Gamma_1 \) and \( \Gamma_2 \) be given by (1)-(2) and satisfying (8) and (9). Assume that the functions \( f \) and \( g \) satisfy (4), (5), (6), (10) and (11). Then for initial data \((u_0, u_1, \theta_0) \in H\), the system (3) has a unique (weak) solution \((u, \theta)\) satisfying
\[
(u, u', \theta) \in C([0, \infty); H). \tag{12}
\]

The main result of our paper is the next theorem

**Theorem 2.2** Let \( \Gamma_1 \) and \( \Gamma_2 \) given by (1), (2) and satisfying (8) and (9). Assume that the functions \( f \) and \( g \) satisfy (10), (11) and the inequalities
\[
g(x) \cdot x \geq m_g |x|^2 \forall x \in \mathbb{R}^n \quad |x| \geq 1, \tag{13}
\]
\[
|x|^2 + |g(x)|^2 \leq G(g(x) \cdot x) \forall x \in \mathbb{R}^n \quad |x| \leq 1, \tag{14}
\]
\[
|x|^2 + |f(x)|^2 \leq G(f(x) \cdot x) \forall x \in \mathbb{R}^n \quad |x| \leq 1, \tag{15}
\]

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where $m_g$ is a positive constant and $G$ a concave function defined on $\mathbb{R}_+$ such that $G(0)=0$. Then there exist positive constants $\tau$, $r_1$, $r_2$ and a time $T_1 > 0$ (depending on $\tau$, $E(0)$, $|\Gamma_2|$, $|\Omega|$) such that the energy of any solution of (3) satisfies

$$E(t) \leq r_2 G\left(\Psi^{-1}(r_1 t) r_1 \tau t\right), \quad \forall t \geq T_1,$$

where $\Psi$ is given by

$$\Psi(t) = \int_t^1 \frac{1}{\Phi(s)} ds, \quad \text{with} \quad \Phi(s) = \tau R_1 G^{-1}\left(\frac{s}{r_2}\right) \text{ and } R_1 = \min(|\Gamma_2|, |\Omega|).$$

Explicit decays are presented in Section 4.

**Remark 2.1** The previous theorem still hold if $f = 0$ and $g$ satisfies the previous hypotheses (case of boundary feedback only) or conversely if $\Gamma_2 = \emptyset$ and $f$ satisfies the previous hypotheses (case of internal feedback).

### 3 Well-posedness of the problem

In this Section we prove Theorem 2.1 by reducing system (3) to a first order evolution equation. Let us define the operators $A : (H^1_{\Gamma_1}(\Omega))^n \longrightarrow [(H^1_{\Gamma_1}(\Omega))^n]'$ and $A_0 : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$< Au, v > = \int_{\Omega} [\mu \nabla u \cdot \nabla v + (\lambda + \mu) \text{div} u \text{div} v] dx, \forall u, v \in (H^1_{\Gamma_1}(\Omega))^n,$$

$$< A_0 u, v > = \int_{\Omega} \nabla u \cdot \nabla v dx, \forall u, v \in H^1_0(\Omega).$$

We further introduce the nonlinear operator $B_0$ from $(H^1_{\Gamma_1}(\Omega))^n$ to $[(H^1_{\Gamma_1}(\Omega))^n]'$ by

$$< B_0 u, v > = \int_{\Gamma_2} m \cdot \nu g(u) \cdot v d\Gamma + \int_{\Omega} f(u) \cdot v dx, \forall u, v \in (H^1_{\Gamma_1}(\Omega))^n.$$

**Lemma 3.1** If the functions $f$ et $g$ satisfy (10) and (11), then the operator $B_0$ is well defined.

The proof of this lemma is similar to the one of lemma 3.1 of [5] (see also section 6 of [7]).
To obtain the abstract formulation of (3), we multiply the first identity of the system (3) by \( v \in (H^1_{\Gamma_1}(\Omega))^n \) and we integrate by parts on \( \Omega \), this yields
\[
0 = \int_\Omega [u'' - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u + \alpha \nabla \theta + f(u')] \cdot v \, dx
\]
\[
= \int_\Omega u'' \cdot v \, dx - \mu \int_\Gamma \frac{\partial u}{\partial \nu} \cdot v \, d\Gamma - (\lambda + \mu) \int_\Gamma v \cdot \text{div} u \, d\Gamma
\]
\[
+ \int_\Omega [(\mu \nabla u \nabla v + (\lambda + \mu) \text{div} u \text{div} v)] \, dx + \int_\Omega (\alpha \nabla \theta \cdot v) \, dx + \int_\Omega f(u') \cdot v \, dx
\]
\[
= \int_\Omega u'' \cdot v \, dx + \int_\Omega (\mu \nabla u \nabla v + (\lambda + \mu) \text{div} u \text{div} v) \, dx
\]
\[
+ \int_\Omega \alpha \nabla \theta \cdot v \, dx + \int_\Omega f(u') \cdot v \, dx
\]
\[
= < u'', v > + < Au, v > + < B_0 u', v > + < \alpha \nabla \theta, v > .
\]
This leads to the identity
\[
u'' + Au + B_0 u' + \alpha \nabla \theta = 0.
\]

In a similar manner, if we multiply the second identity of system (3) by \( v \in (H^1_{\Gamma_1}(\Omega))^n \) and if we integrate by parts on \( \Omega \), we obtain
\[
\theta' + A_0 \theta + \beta \text{div} (u') = 0.
\]
Setting
\[
\Phi = (u, u', \theta)
\]
and
\[
\mathcal{A} \Phi = (-u', Au + B_0 u' + \alpha \nabla \theta, A_0 \theta + \beta \text{div} (u')) ,
\]
the system (3) reduce to
\[
\begin{aligned}
\Phi' + \mathcal{A} \Phi &= 0, \\
\Phi(0) &= (u_0, u_1, \theta_0).
\end{aligned}
\]

Lemma 3.2 Under the hypotheese (4), (5), (6), (8), (10) and (11), the operator \( \mathcal{A} \) defined on \( \mathcal{H} \) by (18) with domain
\[
D(\mathcal{A}) = \{ (u, v, \theta) \in \mathcal{H} : v \in (H^1_{\Gamma_1})^n, Au + B_0 v \in (L^2(\Omega))^n, \theta \in H^2(\Omega) \cap H^1_0(\Omega) \}
\]
is maximal monotone. Moreover, \( D(\mathcal{A}) \) is dense in \( \mathcal{H} \).

The proof of this lemma is similar to the one of lemma 3.2 of [5]. The theory of nonlinear semi-groups ( see [12] for example) leads to Theorem 2.1. Thus the energy of the solution of (3) is given by
\[
E(t) = E(u, \theta, t) = \frac{1}{2} \|(u(t), u'(t), \theta(t))\|_{\mathcal{H}}^2.
\]
4 Proof of Theorem 2.2

Deriving (7) in time and integrating by parts in space, we readily see that

$$E'(t) = -\frac{\alpha}{\beta} \int_\Omega |\nabla \theta(x, t)|^2 \, dx - \int_{\Gamma_2} m \cdot \nu g(u'(t)) \cdot u'(t) - \int_\Omega f(u'(t)) \cdot u'(t) \, dx,$$

and consequently

$$E(T) - E(S) = -\frac{\alpha}{\beta} \int_S^T \int_\Omega |\nabla \theta|^2 \, dx \, dt$$

$$- \int_S^T \int_{\Gamma_2} m \cdot \nu g(u'(t)) \cdot u'(t) \, dx \, dt$$

$$- \int_S^T \int_\Omega f(u'(t)) \cdot u'(t) \, d\Gamma \, dt, \forall 0 \leq S \leq T < \infty.$$

The hypotheses (4), (5) and (6) lead to the decay of the energy.

Under additional hypotheses on $f$ and $g$, we will now obtain different types of decay. For that purpose introduce the constant

$$R_0 = \max_{x \in \Omega} \left( \sum_{k=1}^n (x_k - x_{0k})^2 \right)^{1/2},$$

$$R_1 = \min(|\Gamma_2|, |\Omega|),$$

$$K(a) = \max_{x \in \Gamma_2} \frac{2R_0^2 a(x)}{\mu} + (2 - n).$$

Further let $\gamma$ and $\lambda_0$ be the smallest positive constants such that for all $u \in (H^1_{\Gamma_1}(\Omega))^n$

$$\int_{\Gamma_2} |u|^2 \, d\Gamma \leq \gamma^2 \left( \int_\Omega \{ \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} \, u|^2 \} \, dx + \int_{\Gamma_2} am \cdot \nu |u|^2 \, d\Gamma \right),$$

(21)

and

$$\|u\|_{L^2(\Omega))^n} \leq \lambda_0^2 \left( \int_\Omega \{ \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} \, u|^2 \} \, dx + \int_{\Gamma_2} am \cdot \nu |u|^2 \, d\Gamma \right),$$

(22)

respectively.

To prove Theorem 2.2, we are reduced to check the sufficient conditions of Theorem 5.3 of [7]. In our case it remains to show that the linear system associated with (3) is exponentially stable. This system takes the form
Energy decay of a thermoelastic system with nonlinear feedback

\[
\begin{aligned}
&u'' - \mu\Delta u - (\lambda + \mu)\nabla \text{div } u + \alpha \nabla \theta + u' = 0 \text{ in } Q, \\
&\theta' - \Delta \theta + \beta \text{div } u' = 0 \text{ in } Q, \\
&u = 0 \text{ on } \Gamma, \\
&\mu \partial_\nu u + (\lambda + \mu)\nabla \text{div } u\nu + \alpha \nabla \theta + m \cdot \nu u' = 0 \text{ on } \Gamma_2, \\
&u(., 0) = u_0, \quad u'(., 0) = u_1, \quad \theta(., 0) = \theta_0 \quad \text{in } \Omega.
\end{aligned}
\]  

(23)

We start with technical lemma

**Lemma 4.1** For all \(\varepsilon_0 > 0\) and \(T > 0\), there exists a positive constant \(C(\varepsilon_0)\) such that for all \((u, u', \theta)\) solution of (23)

\[
\int_{\Sigma_2T} \alpha m \cdot \nu |u|^2 d\Sigma \leq C(\varepsilon_0)E(0) + \varepsilon_0 \int_0^T E(t) dt.
\]

**Proof:** We proceed as in [1]. For \(t \geq 0\), consider the solution

\[
\begin{aligned}
&-\mu \Delta z - (\lambda + \mu)\nabla \text{div } z = 0 \text{ in } \Omega, \\
&z = u \text{ on } \Gamma.
\end{aligned}
\]  

(24)

this solution is characterized by \(z = \omega + u\) where \(\omega \in (H^1_0(\Omega))^n\) is the unique solution of

\[
\int_{\Omega} (\mu \nabla \omega \nabla v + (\lambda + \mu)\text{div } \omega \text{div } v) dx dt = -\int_{\Omega} (\mu \nabla u \nabla v + (\lambda + \mu)\text{div } u \text{div } v) dx dt,
\]

\[\forall v \in (H^1_0(\Omega))^n.\]

this identity means that

\[
\int_{\Omega} (\mu \nabla u \nabla z dx dt + (\lambda + \mu)\text{div } u \text{div } z) dx dt = \int_{\Omega} (\mu |\nabla z|^2 dx dt + (\lambda + \mu)|\text{div } z|^2) dx dt \geq 0.
\]

(25)

Moreover by Korn’s inequality we have

\[
\int_{\Omega} |z|^2 dx \leq C_0 \int_{\Gamma} |u|^2 d\Sigma
\]

(26)

and

\[
\int_{\Omega} |z'|^2 dx \leq C_0' \int_{\Gamma} |u'|^2 d\Sigma \leq C_0' \int_{\Gamma} m \cdot \nu |u'|^2 d\Sigma
\]

(27)

where \(C_0, C_0'\) are positive constants.

For \(0 < T < \infty\), we set

\[
Q_T = \Omega \times [0, T], \\
\Sigma_T = \Gamma \times [0, T]; \quad \Sigma_{1T} = \Gamma_1 \times [0, T]; \quad \Sigma_{2T} = \Sigma_T \setminus \Sigma_{1T}.
\]
Multiplying the first identity of (23) by $z$ and integrating on $Q_T$ we obtain
\[ \int_{Q_T} z(u'' - \mu \Delta u - (\lambda + \mu) \nabla \cdot u + \alpha \nabla \theta + u') \, dx \, dt = 0. \]

Applying Green’s formula and taking into account the boundary conditions in (23) and (24), we get
\[ \int_{Q_T} (zu'' + \mu \nabla u \nabla z + (\lambda + \mu) \nabla \cdot u z + \alpha z \nabla \theta + u' z) \, dx \, dt + \int_{\Sigma_T} a m \cdot \nu |u|^2 \, d\Sigma + \int_{\Sigma_T} m \cdot \nu u' u' \, d\Sigma = 0. \]

Integrating by parts in $t$ and using (25), we obtain
\[ - \int_{\Sigma_T} a m \cdot \nu |u|^2 \, d\Sigma \leq - \int_{\Sigma_T} m \cdot \nu u' u' \, d\Sigma + \int_{Q_T} z' u' \, dx \, dt \]
\[ - \alpha \int_{Q_T} z \nabla \theta \, dx \, dt - \int_{Q_T} u' z \, dx \, dt - \int_{\Omega} z u' \big|_{t=0}. \]

Fix an arbitrary $\varepsilon_0 > 0$. Using several times (20) (with $f(x) = g(x) = x$), (26), (27) and Young’s inequality, we can estimate the different integrals of the right-hand side of the above inequality as follows:
\[ - \int_{\Sigma_T} m \cdot \nu u' u' \, d\Sigma \leq \varepsilon_0 \int_{\Sigma_T} m \cdot \nu |u|^2 \, d\Sigma + \frac{1}{4 \varepsilon_0} \int_{\Sigma_T} m \cdot \nu |u'|^2 \, d\Sigma \]
\[ \leq 2 \varepsilon_0 R_0 \gamma^2 \int_0^T E(t) \, dt + \frac{1}{4 \varepsilon_0} E(0), \]
\[ - \int_{Q_T} z' u' \, dx \, dt \leq \varepsilon_0 \int_{Q_T} |u'|^2 \, dx \, dt + \frac{1}{4 \varepsilon_0} \int_{Q_T} |z'|^2 \, dx \, dt \]
\[ \leq \varepsilon_0 \int_0^T E(t) \, dt + \frac{C_0'}{4 \varepsilon_0} E(0), \]
\[ - \int_{Q_T} \alpha \nabla \theta \cdot z \, dx \, dt \leq \frac{\alpha^2}{4 \varepsilon_0} \int_{Q_T} |\nabla \theta|^2 \, dx \, dt + \varepsilon_0 \int_{Q_T} |z|^2 \, d\Sigma \]
\[ \leq \frac{\alpha \beta}{4 \varepsilon_0} E(0) + 2 \varepsilon_0 C_0 \gamma^2 \int_0^T E(t) \, dt, \]
\[
\int_{Q_T} u'z \, dx \, dt \leq \varepsilon_0 \int_{Q_T} |z|^2 \, dx \, dt + \frac{1}{4\varepsilon_0} \int_{Q_T} |u'|^2 \, d\Sigma
\]
\[
\leq \varepsilon_0 C_0 \gamma^2 \int_0^T E(t) \, dt + \frac{1}{4\varepsilon_0} E(0),
\]
\[
\int_{\Omega} z u'|^T_0 \leq 4(1 + C_0 \gamma^2) E(0).
\]

Using these different estimates, we arrive at the requested estimate.

**Proof of Theorem 2.2:** Let us introduce the following constant
\[
c_1 = \alpha \beta (n-1)^2 + 4\alpha \beta R_0^2,
\]
\[
N = \lambda_0^2 + \frac{2}{\mu} + \frac{\gamma^2 \lambda_0^2 R_0^2(n-1)^2}{c_1} + 4 \frac{R_0^2}{\mu c_1}.
\]

Fix \( \varepsilon > 0 \) such that
\[
0 < \varepsilon < \frac{2}{1+N}
\]
and define the constant
\[
k_1 = 1 + \frac{2R_0^2}{\mu} + \frac{c_1}{4\varepsilon},
\]
\[
k_2 = \frac{\alpha \beta (n-1)^2}{4\varepsilon} + \frac{\alpha \beta R_0^2}{\varepsilon} + \lambda_0^2.
\]

Multiplying the first identity of (23) by \( M_i = 2m_k \frac{\partial u_i}{\partial x_k} + (n-1)u_i \) and integrating by parts on \( Q_T \) (the convention of repeated indices is adopted), we obtain
\[
\int_{Q_T} u''_i M_i \, dx \, dt
\]
\[
= (u'_i(t), 2m_k \frac{\partial u_i}{\partial x_k})|_0^T - \int_{\Sigma_T} m_k \nu_k |u'_i|^2 \, d\Sigma + n \int_{Q_T} |u'_i|^2 \, dx \, dt
\]
\[
+ (n-1)(u'_i, u_i)|_0^T - (n-1) \int_{Q_T} |u'_i|^2 \, dx \, dt
\]
\[
= 2(u'_i(t), m_k \frac{\partial u_i}{\partial x_k} + \frac{n-1}{2} u_i)|_0^T - \int_{\Sigma_T} m_k \nu_k |u'_i|^2 \, d\Sigma + \int_{Q_T} |u'_i|^2 \, dx \, dt.
\]
\[ \int_{Q_T} \Delta u_i M_i \, dx \, dt = 2 \int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} \, d\Sigma - \int_{\Sigma_T} m_k \nu_k |\nabla u_i|^2 \, d\Sigma + (n - 2) \int_{Q_T} |\nabla u_i|^2 \, dx \, dt \]

\[ + \, (n - 1) \int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} u_i - (n - 1) \int_{Q_T} |\nabla u_i|^2 \, dx \, dt \]

\[ = 2 \int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} m_k \frac{\partial u_i}{\partial x_k} \, d\Sigma - \int_{\Sigma_T} m_k \nu_k |\nabla u_i|^2 \, d\Sigma + (n - 1) \int_{\Sigma_T} \frac{\partial u_i}{\partial \nu} u_i \]

\[ - \int_{Q_T} |\nabla u_i|^2 \, dx \, dt. \]

\[ \int_{Q_T} \frac{\partial}{\partial x_i} (\text{div } u) M_i \, dx \, dt = 2 \int_{\Sigma_T} \text{div } m_k \frac{\partial u_i}{\partial x_k} \nu_i \, d\Sigma \]

\[ - \int_{\Sigma_T} m_k \nu_k |\text{div } u|^2 \, d\Sigma + (n - 2) \int_{Q_T} |\text{div } u|^2 \, dx \, dt \]

\[ + \, (n - 1) \int_{\Sigma_T} \text{div } u_i u_i - (n - 1) \int_{Q_T} |\text{div } u|^2 \, dx \, dt \]

\[ = 2 \int_{\Sigma_T} \text{div } m_k \frac{\partial u_i}{\partial x_k} \nu_i \, d\Sigma - \int_{\Sigma_T} m_k \nu_k |\text{div } u|^2 \, d\Sigma. \]

Using these different identities, we obtain

\[ 2 \int_0^T E(t) \, dt = \int_{\Sigma_T} [u_i'|^2 - \mu |\nabla u_i|^2 - (\lambda + \mu) |\text{div } u|^2] m \cdot \nu d\Sigma \]

\[ + \, 2 \int_{\Sigma_T} [\mu \frac{\partial u_i}{\partial \nu} + (\lambda + \mu) \text{div } u_i] m_k \frac{\partial u_i}{\partial x_k} \]

\[ + \, (n - 1) \int_{\Sigma_T} [\mu \frac{\partial u_i}{\partial \nu} + (\lambda + \mu) \text{div } u_i] u_i \, d\Sigma \]

\[ - \, 2(u', m_k \frac{\partial u_i}{\partial x_k} + m \frac{1}{2} a u_i')_0^T - 2\alpha \int_{Q_T} \frac{\partial \theta}{\partial x_i} (m_k \frac{\partial u_i}{\partial x_k} + \frac{n}{2} u_i) \, dx \, dt \]

\[ + \, \frac{\alpha}{\beta} \int_{Q_T} \theta^2 \, dx \, dt + \int_{\Sigma_T} a u_i |\text{div } u_i|^2 - 2 \int_{Q_T} u_i' m_k \frac{\partial u_i}{\partial x_k} - (n - 1) \int_{Q_T} u_i u_i' \, dx \, dt \]

Taking into account the boundary conditions (23)( implying in particular
Energy decay of a thermoelastic system with nonlinear feedback

\[
\frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial x_k} \nu_k \text{ sur } \Sigma_{1T},
\]
we arrive at

\[
2 \int_0^T E(t) dt = \sum_{i=1}^7 I_i,
\]
where we have set

\[
I_1 := \int_{\Sigma_{1T}} m \cdot \nu[\mu \frac{\partial u_i}{\partial \nu}]^2 + (\lambda + \mu) |\text{div } u|^2 d\Sigma,
\]
\[
I_2 := \int_{\Sigma_{2T}} m \cdot \nu[u_i]^2 - \mu |\nabla u_i|^2 - (\lambda + \mu) |\text{div } u|^2 d\Sigma,
\]
\[
I_3 := -2 \int_{\Sigma_{2T}} m \cdot \nu[a u_i + u_i'] \frac{\partial u_i}{\partial x_k} d\Sigma,
\]
\[
I_4 := -(n - 1) \int_{\Sigma_{2T}} m \cdot \nu[a u_i + u_i'] u_i d\Sigma + \int_{\Sigma_{2T}} a m \cdot \nu |u_i|^2 d\Sigma,
\]
\[
I_5 := -2(u_i', m \frac{\partial u_i}{\partial x_k} + \frac{n - 1}{2} u_i)'_0 + (n - 1) \int_{Q_T} u_i u_i' dxdt,
\]
\[
I_6 := -2 \alpha \int_{Q_T} \nabla \theta (m \frac{\partial u_i}{\partial x_k} + \frac{n - 1}{2} u_i) dxdt + \frac{\alpha}{\beta} \int_{Q_T} |\theta|^2 dxdt,
\]
\[
I_7 := -2 \int_{Q_T} u_i' m \frac{\partial u_i}{\partial x_k}.
\]

It the remains to estimate each term \( I_i \):

\( I_1 \leq 0 \) since \( m \cdot \nu \leq 0 \) on \( \Sigma_1 \) and also

\[
I_2 \leq \int_{\Sigma_{2T}} m \cdot \nu(|u'|^2 - \mu |\nabla u|^2) d\Sigma.
\]

Young’s inequality and definition of \( R_0 \) imply

\[
I_3 \leq 2 \frac{R_0^2}{\mu} \int_{\Sigma_{2T}} m \cdot \nu a^2 u_i^2 + \frac{\mu}{2} \int_{\Sigma_{2T}} m \cdot \nu |\nabla u_i|^2
\]
\[
+ \frac{R_0^2}{\mu} \int_{\Sigma_{2T}} m \cdot \nu |u_i'|^2 + \frac{\mu}{2} \int_{\Sigma_{2T}} m \cdot \nu |\nabla u_i|^2.
\]

Thus we have

\[
I_3 \leq 2 \frac{R_0^2}{\mu} \int_{\Sigma_{2T}} m \cdot \nu a^2 u_i^2 d\Sigma + \frac{2R_0^2}{\mu} \int_{\Sigma_{2T}} m \cdot \nu |u_i'|^2 d\Sigma + \mu \int_{\Sigma_{2T}} m \cdot \nu |\nabla u_i|^2 d\Sigma.
\]
Similarly,
\[ I_4 \leq \frac{c_1}{4\varepsilon} \int_{\Sigma_T} m \cdot \nu |u'_i|^2 d\Sigma + \frac{(n-1)^2}{c_1} \varepsilon \int_{\Sigma_T} m \cdot \nu |u_i|^2 d\Sigma + (2-n) \int_{\Sigma_T} a_m \cdot \nu |u_i|^2 d\Sigma \]
\[ \leq \frac{c_1}{4\varepsilon} \int_{\Sigma_T} m \cdot \nu |u'_i|^2 d\Sigma + (2-n) \int_{\Sigma_T} a_m \cdot \nu |u_i|^2 d\Sigma \]
\[ + \frac{(n-1)^2 \varepsilon^2 R_0^2}{c_1} \int_0^T E(t) dt. \]

The inequalities
\[ |2 \int_{\Omega} u'_i m \cdot \nabla u_i dx| \leq \frac{R_0^2}{\mu^2} ||u'_i(t)||^2 + \mu ||\nabla u_i(t)||^2 \leq \frac{2R_0^2}{\mu^2} E(t), \]
\[ |(n-1) \int_{\Omega} u'_i u_i dx| \leq \frac{n-1}{2} \lambda_0 ||u'_i(t)||^2 + \mu ||u(t)||_{L^2(\Omega)}^2 \leq (n-1) \lambda_0 E(t), \]
\[ \frac{(n-1)}{2} \int_{\Omega} u_i^2 \leq (n-1) \lambda_0^2 E(t), \]
and the definition of \( k_1 \) lead to
\[ I_5 \leq k_1 E(0). \]

By Young’s inequality, the definition of \( R_0 \) and of \( \lambda_0 \), and taking into account (20) (with \( f(x) = g(x) = x \)), we have successively
\[ I_6 \leq \frac{\alpha^2(n-1)^2}{4\varepsilon} \int_{Q_T} |\nabla \theta|^2 + \varepsilon \int_{Q_T} |u|^2 + \frac{\alpha^2 R_0^2}{\varepsilon} \int_{Q_T} |\nabla \theta|^2 \]
\[ + \frac{\alpha}{\beta} \int_{Q_T} |\theta|^2 dx dt + \varepsilon \int_{Q_T} |\nabla u|^2, \]
\[ \leq \frac{\alpha^2(n-1)^2}{4\varepsilon} + \frac{\alpha^2 R_0^2}{\varepsilon} + \lambda_0^2 \int_{Q_T} \frac{\alpha}{\beta} |\nabla \theta|^2 + [\varepsilon \lambda_0^2 + \frac{2\varepsilon}{\mu}] \int_0^T E(t) dt \]
\[ \leq k_2 E(0) + [\varepsilon \lambda_0^2 + \frac{2\varepsilon}{\mu}] \int_0^T E(t) dt. \]

\[ I_7 = -2 \int_{Q_T} u'_i m_k \frac{\partial u_i}{\partial x_k} \leq \frac{c_1}{4\varepsilon} \int_{Q_T} |u'_i|^2 + \frac{4R_0^2 \varepsilon}{\mu c_1} \int_{Q_T} \mu |\nabla u|^2. \]

All together we have
\[ 2 \int_0^T E(t) dt \leq I_9 + k_1 \left( \int_{\Sigma_T} m \cdot \nu |u'_i|^2 d\Sigma + \int_{Q_T} |u'_i|^2 dx dt \right) \]
\[ + k_2 E(0) + \varepsilon N \int_0^T E(t) dt, \]
where we have set

\[ I_9 = \int_{\Sigma_T} \left[ \frac{2R_0^2a^2}{\mu} + (2 - n)a \right] m \cdot \nu |u|^2 d\Sigma. \]

The definition of \( K(a) \) leads to

\[ I_9 \leq K(a) \int_{\Sigma_T} am \cdot \nu |u|^2 d\Sigma. \]

Applying Lemma 4.1 with \( \varepsilon_0 = \frac{\varepsilon}{K(a)} \), there exist a positive constant \( C(\varepsilon) \) such that

\[ I_9 \leq C(\varepsilon) E(0) + \varepsilon \int_0^T E(t) dt. \]

Finally, setting

\[ C_1 = \frac{k_2 + C(\varepsilon)}{2 - \varepsilon(1 + N)}, \quad C_2 = \frac{k_1}{2 - \varepsilon(1 + N)}, \]

we conclude that

\[ \int_0^T E(t) \leq C_1 E(0) + C_2 \left( \int_{\Sigma_T} m \cdot \nu |u'|^2 d\Sigma dt + \int_{Q_T} |u'|^2 dx dt \right). \]

This estimate remains valid for weak solutions by a density argument.

We then conclude by Theorem 5.3 of [7].

To complete the proof, we now define as in formalism of [7] the operators \( A_1 \) and \( \mathcal{I}_U \) associated to (23) as follows: \( A_1 \) is defined on

\[ \mathcal{V} := (H^1_{\Gamma_1})^n \times (H^1_{\Gamma_1})^n \times H^1_0(\Omega) \]

by

\[ A_1 \Phi = (-v, Au + \alpha \nabla \theta, \beta \text{div} v). \]

Taking into account the feedbacks in (23) and identity (11) of [7], we set

\[ U = (L^2(\Gamma_2))^n \times (L^2(\Omega))^n \times L^2(\Omega) \]

and we define the application

\[ I_U : \mathcal{V} \rightarrow U \]

\[ (u, v, \theta) \mapsto (v|_{\Gamma_2}, v, \theta) \]

and \( \mathcal{I}_U \) from \( \mathcal{V} \) to \( \mathcal{V}' \) by

\[ \langle \mathcal{I}_U(u, v, \theta), (u^*, v^*, \theta^*) \rangle = \langle I_U(u, v, \theta), I_U(u^*, v^*, \theta^*) \rangle \]

\[ = \int_{\Gamma_2} m \cdot \nu v v^* d\Gamma + \int_\Omega v v^* dx + \frac{\alpha}{\beta} \int_\Omega \nabla \theta \cdot \nabla \theta^* dx. \]
5 examples

1. If we assume that $f = 0$ and $g$ satisfy (5), (11), (13), (14) as well as

\[ x \cdot g(x) \geq c_0 |x|^{p+1}, \forall |x| \leq 1, \]  \hspace{1cm} (29)

\[ |g(x)| \leq C_0 |x|^{\alpha}, \forall |x| \leq 1, \]  \hspace{1cm} (30)

where $c_0, C_0$ are positive constants, $\alpha \in (0, 1]$ and $p \geq \alpha$ then making the choice

\[ G(x) = x^{\frac{q}{p+1}} \text{ and } q = \frac{p + 1}{\alpha} - 1 \]

we obtain decays similar to the ones from Theorem 2.3 to [5]. Indeed, if $p = \alpha = 1$, then $\Psi^{-1}(t) = e^{-t}$ and we conclude and exponential decay. Conservely, if $p + 1 \geq 2\alpha$, then $\Psi^{-1}(t) = t^{\frac{p+1}{2\alpha}}$ and we obtain a decay of order $t^{\frac{p+1}{2\alpha}}$.

2. In a similar manner as in examples 5.6 et 5.8 of [7] good choices of $f$ and $g$ allow to obtain logarithmic, double logarithmic decay etc..

References


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