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Effect of Magnetic Field on Torsional Surface Waves in Non-Homogeneous Viscoelastic Medium

Rajneesh Kakar*
Principal
DIPS Polytechnic College, Hoshiarpur-146001, India
* Corresponding author. Tel.: +919915716560; fax: +911886237166
E-mail address: rkakar_163@rediffmail.com

Monika Rani
Research Scholar
Punjab Technical University, Jalandhar, 144601, India

K. C. Gupta
Supervisor
Punjab Technical University, Jalandhar, 144601, India

Abstract

This paper deals with the propagation of magnetoelastic torsional waves in non-homogeneous viscoelastic cylindrically aeolotropic material. The elastic constants and non-homogeneity in density of the material are in the form $\delta_{ij} = C_{ij} r^{l}$ and $\rho = \rho_0 r^{m}$ respectively, where C_{ij}, ρ_0 are constants; r is radius vector; *l* and *m* are any integers. Frequency equation in each case has been derived and the graphs have been plotted showing the effect of variation of elastic constants and the presence of magnetic field. It is observed that the torsional elastic waves in a viscoelastic solid body propagating under the influence of a superimposed magnetic field can be different significantly from that of those propagating in the absence of a magnetic field. The numerical calculations have been presented graphically by using MATLAB.

Mathematics Subject Classification: 74J15, 74Dxx, 74E05

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1 Introduction

A large amount of literature is available on surface wave in the monograph of Ewing [1]. But there is a very few problems of cylindrically aeolotropic elastic material have been considered so far because of the inherent difficulty in solving complicated simultaneous partial differential equations. Kaliski [2], Narain [3] and many others have investigated the magnetoelastic torsional surface waves. White [4] has investigated cylindrical waves in transversely isotropic media. The elastic cylindrical shell under radial impulse was studied by Mcivor [5]. Cinelli [6] has investigated dynamic vibrations and stresses in elastic cylinders and spheres. Pan and Heyliger [7] have given the exact solutions for magneto-electro-elastic Laminates in cylindrical bending. The wave propagation in non-homogeneous magneto-electro-elastic plates has been solved by Bin et al. [8]. Kong et al. [9] solved the problem of thermo-magneto-dynamic stresses and perturbation of magnetic field vector in non-homogeneous hollow cylinder. Recently, Kakar et al. [10], [11] and [12] studied various surface waves in elastic as well in viscoelastic media.
In this study, the torsional surface waves are investigated in non-homogeneous viscoelastic cylindrically aeolotropic material subjected to a magnetic field. The problem is solved analytically by using Bessel's functions and numerically by using MATLAB.

2 Basic equations

The problem is dealing with magnetoelasticity. Therefore the basic equations will be electromagnetism and elasticity. Therefore, the Maxwell equations of electromagnetic field in the absence of the displacement current (in system-international unit) are [14]

$$\vec{\nabla}\cdot\vec{\mathrm{E}}=0, \tag{1a}$$
$$\vec{\nabla}\cdot\vec{\mathrm{B}}=0 \ , \tag{1b}$$
$$\vec{\nabla}\times\vec{\mathrm{E}}=-\frac{\overrightarrow{\partial \mathrm{B}}}{\partial t} \ , \tag{1c}$$
$$\vec{\nabla}\times\vec{\mathrm{B}}= \ \mu_0\varepsilon_0\frac{\overrightarrow{\partial \mathrm{E}}}{\partial t}. \tag{1d}$$

Where, $\vec{\mathrm{E}}$, $\vec{\mathrm{B}}$, μ_0 and ε_0 are electric field, magnetic field induction, permeability and permittivity of the vacuum. For vacuum, $\mu_0 = 4\pi\times10^{-7}$ and $\varepsilon_0 = 8.85\times10^{-12}$ in SI units. Also, the term Ohm's law is

$$\vec{J}=\sigma\vec{E}, \tag{2a}$$

Where, $\vec{J}$ is the current density and σ is a material conductivity.
The Lorentz force on the charge carriers is [14]

$$\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B}) = \sigma(\vec{E} + \frac{\partial v}{\partial t} \times \vec{B}). \tag{2b}$$

The homogeneous form of the electromagnetic wave equation is [14]

$$\left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{\mathrm{E}} = 0, \tag{3a}$$

$$\left(\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right) \vec{\mathrm{B}} = 0. \tag{3b}$$

Where, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

The dynamical equations of motion in cylindrical coordinate (r, θ, z) are [13], [18]

$$\frac{\partial s_{rr}}{\partial r} + \frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta} + \frac{\partial s_{rz}}{\partial z} + \frac{1}{r}(s_{rr} - s_{\theta\theta}) + T_R = \rho \frac{\partial^2 u}{\partial t^2}, \tag{4a}$$

$$\frac{\partial s_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta} + \frac{\partial s_{\theta z}}{\partial z} + \frac{2 s_{r\theta}}{r} + T_\theta = \rho \frac{\partial^2 v}{\partial t^2}, \tag{4b}$$

$$\frac{\partial s_{rz}}{\partial r} + \frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta} + \frac{\partial s_{zz}}{\partial z} + \frac{s_{rz}}{r} + T_Z = \rho \frac{\partial^2 w}{\partial t^2}. \tag{4c}$$

Where, $s_{rr}, s_{r\theta}, s_{rz}, s_{rr}.s_{\theta\theta}, s_{\theta z}, s_{zz}$ are the respective stress components, T_R, T_θ, T_Z are the respective body forces and u, v, w are the respective displacement components.
The stress-strain relations are [18]

$$s_{rr} = \delta_{11}^0 e_{rr} + \delta_{12}^0 e_{\theta\theta} + \delta_{13}^0 e_{zz}, \tag{5a}$$

$$s_{\theta\theta} = \delta_{21}^0 e_{rr} + \delta_{22}^0 e_{\theta\theta} + \delta_{23}^0 e_{zz}, \tag{5b}$$

$$s_{zz} = \delta_{31}^0 e_{rr} + \delta_{32}^0 e_{\theta\theta} + \delta_{33}^0 e_{zz}, \tag{5c}$$

$$s_{rz} = \delta_{44}^0 e_{rz}, \tag{5d}$$

$$s_{\theta z} = \delta_{55}^0 e_{\theta z}, \tag{5e}$$

$$s_{r\theta} = \delta_{66}^0 e_{r\theta}. \tag{5f}$$

Where, $\delta_{ij} =$ elastic constants (ij = 1, 2……6).
The elastic constants of viscoelastic medium are [21]

$$\delta_{ij}^0 = \delta_{ij} + \delta_{ij}^{/} \frac{\partial}{\partial t} + \delta_{ij}^{//} \frac{\partial^2}{\partial t^2} \; (ij = 1, 2 \ldots\ldots 6). \tag{6}$$

Where, $\delta_{ij}^{/}$ and $\delta_{ij}^{//}$ are the first and second order derivatives of δ_{ij}.

The strain components are [20]

$$e_{rr}=\frac{1}{2}\frac{\partial u}{\partial r}, \tag{7a}$$

$$e_{\theta\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}\right), \tag{7b}$$

$$e_{zz}=\frac{1}{2}\frac{\partial w}{\partial z}, \tag{7c}$$

$$e_{\theta z}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial v}{\partial z}\right), \tag{7d}$$

$$e_{rz}=\frac{1}{2}\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right), \tag{7e}$$

$$e_{zz}=\frac{1}{2}\frac{\partial w}{\partial z}, \tag{7f}$$

The rotational components are [20]

$$\Omega_r=\frac{1}{2}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}-\frac{\partial v}{\partial z}\right), \tag{8a}$$

$$\Omega_\theta=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u}{\partial z}-\frac{\partial w}{\partial r}\right), \tag{8b}$$

$$\Omega_z=\frac{1}{r}\left(\frac{\partial (rv)}{\partial r}-\frac{\partial u}{\partial \theta}\right). \tag{8c}$$

Equations for the propagation of small elastic disturbances in a perfectly conducting viscoelastic solid will have the body force in terms of electromagnetic force $\left(\vec{J}\times\vec{B}\right)$ (using Eq. (4)) and are

$$\frac{\partial s_{rr}}{\partial r}+\frac{1}{r}\frac{\partial s_{r\theta}}{\partial \theta}+\frac{\partial s_{rz}}{\partial z}+\frac{1}{r}(s_{rr}-s_{\theta\theta})+\left(\vec{J}\times\vec{B}\right)_R=\rho\frac{\partial^2 u}{\partial t^2}, \tag{9a}$$

$$\frac{\partial s_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta\theta}}{\partial \theta}+\frac{\partial s_{\theta z}}{\partial z}+\frac{2s_{r\theta}}{r}+\left(\vec{J}\times\vec{B}\right)_\theta=\rho\frac{\partial^2 v}{\partial t^2}, \tag{9b}$$

$$\frac{\partial s_{rz}}{\partial r}+\frac{1}{r}\frac{\partial s_{\theta z}}{\partial \theta}+\frac{\partial s_{zz}}{\partial z}+\frac{s_{rz}}{r}+\left(\vec{J}\times\vec{B}\right)_Z=\rho\frac{\partial^2 w}{\partial t^2}. \tag{9c}$$

Let us assume the components of magnetic field intensity $\vec{H}$ are $H_r=H_\theta=0$ and $H_z=H$ constant. Therefore, the value of magnetic field intensity is

$$\vec{H}(0,0,H)=\vec{H}_0+\vec{H}_i \tag{10}$$

Where, $\vec{H}_0$ is the initial magnetic field intensity along z-axis and $\vec{H}_i$ is the perturbation in the magnetic field intensity.

The relation between magnetic field intensity $\vec{H}$ and magnetic field induction $\vec{B}$ is

$$\vec{B}=\mu_0\vec{H} \tag{11}$$

(For vacuum, $\mu_0 = 4\pi\times10^{-7}$ SI units.)

From Eq. (1), Eq. (2), Eq. (3) and Eq. (10), we get

$$\nabla^2\vec{H} = \mu_0\sigma\left\{\frac{\partial\vec{H}}{\partial t} + \vec{\nabla}\times\left(\frac{\partial v}{\partial t}\times\vec{H}\right)\right\} \tag{12}$$

The components of Eq. (12) can be written as

$$\frac{\partial H_r}{\partial t} = \frac{1}{\mu_0\sigma}\nabla^2 H_{r,} \tag{13a}$$

$$\frac{\partial H_\theta}{\partial t} = \frac{1}{\mu_0\sigma}\nabla^2 H_{\theta,} \tag{13b}$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0\sigma}\nabla^2 H. \tag{13c}$$

3 Formulation of the problem

Let us consider a semi-infinite hollow cylindrical tube of radii **α** and **β**. Let the elastic properties of the shell are symmetrical about z-axis, and the tube is placed in an axial magnetic field surrounded by vacuum. Since, we are investigating the torsional waves in an aeolotropic cylindrical tube therefore the displacement vector has only v component. Hence,

$$u = 0, \tag{14a}$$

$$w = 0 \tag{14b}$$

$$v = v(r, z). \tag{14c}$$

Therefore, from Eq. (14) and Eq. (7), we get,

$$e_{rr} = e_{\theta\theta} = e_{zz} = e_{zr} = 0, \tag{15a}$$

$$e_{\theta z} = \frac{1}{2}\left(\frac{\partial v}{\partial z}\right), \tag{15b}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right). \tag{15c}$$

From Eq. (14) and Eq. (8), we get,

$$\Omega_r = \frac{1}{2}\left(\frac{\partial v}{\partial z}\right), \tag{16a}$$

$$\Omega_\theta = 0, \tag{16b}$$

$$\Omega_z = \frac{\partial v}{\partial r} + \frac{v}{r}. \tag{16c}$$

Using Eq. (14), Eq. (15) and Eq. (6), the Eq. (5) becomes

$$s_{rr} = s_{\theta\theta} = s_{zz} = s_{rz} = 0, \tag{17a}$$

$$s_{r\theta} = (\delta_{66} + \delta_{66}^{/}\frac{\partial}{\partial t} + \delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r} - \frac{v}{r}), \tag{17b}$$

$$s_{\theta z}=(\delta_{55}+\delta_{55}^{/}\frac{\partial}{\partial t}+\delta_{55}^{//}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r}). \tag{17c}$$

Where, $\delta_{ij}^{/}$ and $\delta_{ij}^{//}$ are the first and second order derivatives of δ_{ij}.

For perfectly conducting medium, (i.e. $\sigma\to\infty$), it can be seen that Eq. (2) becomes

$$\vec{\mathrm{E}}=\left[-\frac{\mu_0\mathrm{H}}{c}\frac{\partial v}{\partial t},0,0\right] \tag{18}$$

Eq. (1) and Eq. (18), the Eq. (13) becomes,

$$\vec{\mathrm{H}}_i=\left[0,\mathrm{H}\frac{\partial v}{\partial z},0\right] \tag{19}$$

From the above discussion, the electric and magnetic components in the problem are related as

$$\left[-\frac{\mu_0\mathrm{H}}{c}\frac{\partial v}{\partial t},0,0\right]=\left[0,\mathrm{H}\frac{\partial v}{\partial z},0\right] \tag{20}$$

Using Eq. (19) and Eq. (1) to get the components of body force in terms of Gaussian system of units as:

$$\mathrm{T}=\left[0,\frac{\mathrm{H}}{4\pi}\frac{\partial^2 v}{\partial z^2},0\right] \tag{21}$$

Eq. (17) and Eq. (20) satisfy the Eq. (4a) and Eq. (4c), therefore, the remaining

$$\left\{\begin{aligned}&\frac{\partial}{\partial r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})+\frac{\partial}{\partial z}(\delta_{55}+\delta_{55}^{/}\frac{\partial}{\partial t}+\delta_{55}^{//}\frac{\partial^2}{\partial t^2})(-\frac{1}{2}\frac{\partial v}{\partial r})\\&+\frac{2}{r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})-\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2}\end{aligned}\right\}=\rho\frac{\partial^2 v}{\partial t^2} \tag{22}$$

Eq. (4b) becomes

Let $C_{ij}=\delta_{ij}r^l, C_{ij}^{/}=\delta_{ij}^{/}r^l, C_{ij}^{//}=\delta_{ij}^{//}r^l$ and $\rho=\rho_0 r^m$ (23)

Where, δ_{ij}, $\delta_{ij}^{/}$, $\delta_{ij}^{//}$ and ρ_0 are constants, r is the radius vector and l,m are non-homogeneities.

From Eq. (23), we get Eq. (17) as

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r}), \tag{24a}$$

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r}), \tag{24b}$$

Using Eq. (23), the Eq. (22) becomes

$$\left\{\begin{aligned}&\frac{\partial}{\partial r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})+\frac{\partial}{\partial z}(\delta_{55}+\delta_{55}^{/}\frac{\partial}{\partial t}+\delta_{55}^{//}\frac{\partial^2}{\partial t^2})r^l(-\frac{1}{2}\frac{\partial v}{\partial r})\\&+\frac{2}{r}(\delta_{66}+\delta_{66}^{/}\frac{\partial}{\partial t}+\delta_{66}^{//}\frac{\partial^2}{\partial t^2})r^l\frac{1}{2}(\frac{\partial v}{\partial r}-\frac{v}{r})-\frac{H^2}{4\pi}\frac{\partial^2 v}{\partial z^2}\end{aligned}\right\}=\rho_0 r^m\frac{\partial^2 v}{\partial t^2} \tag{25}$$

4 Solution of the problem

Let $v = \xi(r)e^{i(\varsigma z+\zeta t)}$ [16] be the solution of Eq. (25). Hence, Eq. (25) reduces to

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{(l+1)}{r}\frac{\partial \xi}{\partial r} - \frac{(l+1)}{r^2}\xi + \Theta_1^2 \xi + \Theta_2^2 \frac{\xi}{r^l} = 0 \tag{26}$$

Where,

$$\Theta_1^2 = \frac{2\rho_0 \zeta^2 - (\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2)\varsigma^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2}, \tag{27a}$$

$$\Theta_2^2 = \frac{H^2 \varsigma^2}{2\pi(\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2)}. \tag{27b}$$

Eq. (26) is in complex form, therefore we generalize its solution for $l = 0$ and $l = 2$

4.1 Solution for $l = 0$

For, $l = 0$ the Eq. (26) becomes,

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + (\Xi^2 - \frac{1}{r^2})\xi = 0 \tag{28}$$

Where, (29)

$$\Xi^2 = \Theta_1^2 + \Theta_2^2$$

The solution of Eq. (28) is

$$v = \{PJ_1(Gr) + QX_1(Gr)\}e^{i(\varsigma z+\zeta t)} \tag{30}$$

From Eq. (24) and Eq. (30)

$$s_{r\theta} = \{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2\}\left[\frac{P}{2}\{GJ_0(Gr) - \frac{2}{r}J_1(Gr) + \frac{Q}{2}\{GX_0(Gr) - \frac{2}{r}X_1(Gr)\right]e^{i(\varsigma z+\zeta t)} \tag{31}$$

5 Boundary conditions and frequency equation

The boundary conditions that must be satisfied are

B1. For $r = \boldsymbol{\alpha}$, ($\boldsymbol{\alpha}$ is the internal radius of the tube)

$$s_{r\theta} + \tau_{r\theta} = \tau_{(r\theta)_0}$$

B2. For $r = \boldsymbol{\beta}$, ($\boldsymbol{\beta}$ is the external radius of the tube)

$$s_{r\theta} + \tau_{r\theta} = \tau_{(r\theta)_0}$$

Where $\tau_{r\theta}$ and $\tau_{(r\theta)_0}$ are the Maxwell stresses in the body and in the vacuum, respectively. There will be no impact of these Maxwell stresses. Hence,

$$\tau_{r\theta} = \tau_{(r\theta)_0} = 0 \tag{32}$$

On simplification, Eq. (18) and Eq. (30) gives

$$\mathrm{E} = -\frac{\mu_0 \mathrm{H}}{c} i\zeta\{PJ_1(Gr) + QX_1(Gr)\}e^{i(\varsigma z+\zeta t)} \tag{33}$$

$$Let, \mathrm{E}_0 = \Psi e^{i(\varsigma z+\zeta t)}$$

Hence, Eq. (3) becomes

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\gamma^2\Psi=0 \tag{34}$$

Where, $\gamma^2=\frac{\zeta^2}{c^2}-\varsigma^2$ (35)

The solution of the Eq. (34) becomes

$$\Psi=RJ_0(\gamma r)+SX_0(\gamma r) \tag{36}$$

Where J_0 and X_0 are Bessel functions of order zero. R and S are constants.

From Eq. (37) and Eq. (40)

$$\Psi=\{RJ_0(\gamma r)+SX_0(\gamma r)\}e^{i(\varsigma z+\zeta t)} \tag{37}$$

The boundary conditions B1 and B2 with the help of the Eq. (31) and (32) turn into:

$$P\{G\alpha J_0(G\alpha)-2J_1(G\alpha)\}+Q\{G\alpha X_0(G\alpha)-2X_1(G\alpha)\}=0 \tag{38}$$

$$P\{G\beta J_0(G\beta)-2J_1(G\beta)\}+Q\{G\beta X_0(Ga)-2X_1(G\beta)\}=0 \tag{39}$$

Eliminating P and Q from Eq. (38) and Eq. (39)

$$\begin{vmatrix} G\alpha J_0(G\alpha)-2J_1(G\alpha) & G\alpha X_0(G\alpha)-2X_1(G\alpha) \\ G\beta J_0(G\beta)-2J_1(G\beta) & G\beta X_0(Ga)-2X_1(G\beta) \end{vmatrix}=0 \tag{40}$$

On solving Eq. (40), we get the obtained frequency equation

$$\frac{G\alpha J_0(G\alpha)-2J_1(G\alpha)}{G\beta J_0(G\beta)-2J_1(G\beta)}=\frac{G\alpha X_0(G\alpha)-2X_1(G\alpha)}{G\beta X_0(Ga)-2X_1(G\beta)}=0 \tag{41}$$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta=\alpha+\Delta\alpha$ and neglecting $\Delta\alpha^2,\Delta\alpha^3$........, the frequency equation can be written as (Watson [16])

$$\Xi^3\alpha^2+\Xi-1=0 \tag{42}$$

Where,

$$\Xi^2=\frac{2\rho_0\zeta^2-(\delta_{55}+\delta_{55}^{/}i\zeta-\delta_{55}^{//}\zeta^2)\varsigma^2+\frac{H^2}{2\pi}\varsigma^2}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2} \tag{43}$$

Putting the value of Ξ in Eq. (42), the frequency ζ of the wave can be found. Clearly, frequency ζ is dependent on magnetic field.

Put , $\Xi\alpha=\Phi$ (44)

The phase velocity $c_1=\zeta/\varsigma$ can be written as

$$\frac{c_1^2}{c_0^2}=\Phi^2\left(\frac{\lambda}{2\pi\alpha}\right)^2+\mathrm{K}-\frac{\left(\frac{H^2}{4\pi}\right)}{\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2} \tag{45}$$

Where,

$$\lambda = \frac{2\pi}{k},\ \mathrm{K} = \frac{\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2}, \tag{46}$$

$$c_0^2 = \frac{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2}{2\rho_0}$$

The term H i.e. magnetic field is negative in Eq. (45) which reduces the phase velocity of torsional wave.

Case 1

Since the pipe under consideration is made of an aeolotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0 \tag{47}$$

Hence, from Eq. (42), Eq. (44) and Eq. (47) the frequency equation becomes

$$\Phi_0^3 + \Phi_0 - \alpha = 0 \tag{48}$$

Using Eq. (45) and Eq. (46), the phase velocity is

$$c_2^2 = \frac{\delta_{66}}{2\rho_0}\left\{\Phi_0^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + \frac{\delta_{55}}{\delta_{66}} - \frac{H^2}{2\pi\delta_{66}}\right\} \tag{49}$$

$$\Rightarrow \frac{c_2}{c_0} = \left\{\frac{[\frac{\Phi_0}{2\pi}]^2}{[\frac{\alpha}{\lambda}]^2} + \frac{\delta_{55}}{\delta_{66}} - \frac{H^2}{2\pi\delta_{66}}\right\}^{\frac{1}{2}}$$

Where, $c_0^2 = \delta_{66}/2\rho_0$ (50)

The term H i.e. magnetic field is negative in Eq. (49) which reduces the phase velocity of torsional wave. This is in complete agreement with the corresponding classical results [15]

Case 2

If the pipe under consideration is made of an isotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0, \delta_{55} = \delta_{66} = \chi \tag{51}$$

Using Eq. (49) and Eq. (50), the phase velocity is

$$c_2^2 = \frac{\chi}{2\rho_0}\left\{\Phi_0^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + 1 - \frac{H^2}{2\pi\chi}\right\} \tag{52}$$

This is in complete agreement with the corresponding classical results [3]

5.1Solution for l=2

For, $l = 2$ the Eq. (26) becomes,

$$\frac{\partial^2\xi}{\partial r^2} + \frac{3}{r}\frac{\partial\xi}{\partial r} + (\Theta_1^2 - \frac{(3-\Theta_2^2)}{r^2}\xi = 0 \tag{53}$$

Putting $\xi = \frac{1}{r}\mathrm{N}(r)$ in Eq. (53), one get

$$\frac{\partial^2 \mathrm{N}}{\partial r^2}+\frac{1}{r}\frac{\partial \mathrm{N}}{\partial r}+\left[\Theta_1^2-\frac{\mathrm{P}^2}{r^2}\right]\mathrm{N}=0 \tag{54}$$

Where, $\mathrm{P}^2=3-\Theta_2^2$ (55)

Solution of Eq. (54) will be (Watson [16])

$$\mathrm{N}=RJ_{\mathrm{P}}(\Theta_1 r)+SX_{\mathrm{P}}(\Theta_2 r) \tag{56}$$

Putting the value of ξ and N in Eq. (55), we get

$$\mathrm{P}=\frac{1}{r}\{RJ_{\mathrm{P}}(\Theta_1 r)+SX_{\mathrm{P}}(\Theta_1 r)\}e^{i(\varsigma z+\zeta t)} \tag{57}$$

From the Eq. (24) and Eq. (56)

$$s_{r\theta}=(\delta_{66}+\delta_{66}^{/}i\zeta-\delta_{66}^{//}\zeta^2)\begin{bmatrix}\frac{R}{2}\{\Theta_1 rJ_{\mathrm{P}-1}(\Theta_1 r)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1 r)\} \\ +\frac{S}{2}\{\Theta_1 rX_{\mathrm{P}-1}(\Theta_1 r)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1 r)\}\end{bmatrix}e^{i(\varsigma z+\zeta t)}=0 \tag{58}$$

With the help of Eq. (32), Eq. (56) and boundary conditions B1 and B2, we get

$$\frac{R}{2}\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\}+\frac{S}{2}\{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\alpha)\}=0$$
$$\frac{R}{2}\{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\}+\frac{S}{2}\{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\beta)\}=0 \tag{59}$$

Eliminating R and S from Eq. (58) and Eq. (59)

$$\begin{vmatrix}\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\} & \{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\alpha)\} \\ \{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\} & \{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\gamma\beta)\}\end{vmatrix}=0 \tag{60}$$

On solving Eq. (60), we get

$$\frac{\{\Theta_1\alpha J_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\alpha)\}}{\{\Theta_1\alpha X_{\mathrm{P}-1}(\Theta_1\alpha)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\alpha)\}}=\frac{\{\Theta_1\beta J_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)J_{\mathrm{P}}(\Theta_1\beta)\}}{\{\Theta_1\beta X_{\mathrm{P}-1}(\Theta_1\beta)-(\mathrm{P}+2)X_{\mathrm{P}}(\Theta_1\beta)\}} \tag{61}$$

If η_1 is the root of the above equation, then

$$\frac{\{\eta_1 J_{\mathrm{P}-1}(\eta_1)-(\mathrm{P}+2)J_{\mathrm{P}}(\eta_1)\}}{\{\eta_1 X_{\mathrm{P}-1}(\eta_1)-(\mathrm{P}+2)X_{\mathrm{P}}(\eta_1)\}}=\frac{\{\eta_1 F_1 J_{\mathrm{P}-1}(\eta_1 F_1)-(\mathrm{P}+2)J_{\mathrm{P}}(\eta_1 F_1)\}}{\{\eta_1 F_1 X_{\mathrm{P}-1}(\eta_1 F_1)-(\mathrm{P}+2)X_{\mathrm{P}}(\eta_1 F_1)\}} \tag{62}$$

Where, $F_1=\beta/\alpha$

On the theory of Bessel functions, if tube under consideration is very thin i.e. $\beta=\alpha+\Delta\alpha$ and neglecting $\Delta\alpha^2,\Delta\alpha^3$........, the frequency equation can be written as (Watson [18])

$$(\mathrm{P}+2)^2-\left(2\mathrm{P}-1+\frac{1}{\Theta_1}\right)(\mathrm{P}+2)+\Theta_1^2\alpha^2=0 \tag{63}$$

Where,

$$P^2 = 3 - \Theta_2^2 \Rightarrow P^2 = 3 - \frac{H^2\varsigma^2}{2\pi(\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2)}, \tag{64a}$$

$$\Theta_1^2 = \frac{2\rho_0\zeta^2 - (\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2)\varsigma^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2}. \tag{64b}$$

From the Eq. (62), Eq. (63) and Eq. (64), the phase velocity can be written as (same as above Eq. (45) and Eq. (46))

$$\frac{c^2}{c_0^2} = \eta^2\left(\frac{\lambda}{2\pi\alpha}\right)^2 + \frac{\delta_{55} + \delta_{55}^{/} i\zeta - \delta_{55}^{//}\zeta^2}{\delta_{66} + \delta_{66}^{/} i\zeta - \delta_{66}^{//}\zeta^2} \tag{65}$$

Case 1

Since the pipe under consideration is made of an aeolotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0 \tag{66}$$

The frequency equation is given by

$$\frac{\{\Theta_3\alpha J_{P_1-1}(\Theta_3\alpha) - (P+2)J_{P_1}(\Theta_3\alpha)\}}{\{\Theta_3\alpha X_{P_1-1}(\Theta_3\alpha) - (P+2)X_{P_1}(\Theta_3\alpha)\}} = \frac{\{\Theta_3\beta J_{P_1-1}(\Theta_3\beta) - (P+2)J_{P_1}(\Theta_3\beta)\}}{\{\Theta_3\beta X_{P_1-1}(\Theta_3\beta) - (P+2)X_{P_1}(\Theta_3\beta)\}} \tag{67}$$

$$\eta_2^3 + 6\eta_2 - 3\alpha = 0 \tag{68}$$

$$P_1^2 = 3 - \frac{H^2\varsigma^2}{2\pi\delta_{66}}, \qquad \Theta_3^2 = \frac{2\rho_0\zeta^2 - \delta_{55}\varsigma^2}{\delta_{66}}, \qquad \eta_2 = \Theta_3\varsigma \quad at\ P_1 = 1 \tag{69}$$

Using Eq. (65), Eq. (66), Eq. (67) and Eq. (69), we get (calculations are done in the similar manner as for the Eq. (48) to Eq. (50) for $l = 0$ case)

$$\frac{c_3}{c_{01}} = \left[\frac{\left(\frac{\eta_2}{2\pi}\right)^2}{\left(\frac{\alpha}{\lambda}\right)^2} + \frac{\delta_{55}}{\delta_{66}}\right]^{\frac{1}{2}} \tag{70}$$

Where, $c_{01}{}^2 = \delta_{66} / 2\rho_0$

Case 2

If the pipe under consideration is made of an isotropic material, then

$$\delta_{ij}^{/} = \delta_{ij}^{//} = 0, \delta_{55} = \delta_{66} = \chi \tag{71}$$

The frequency equation (calculations are done as for the l=0 case) is

$$\frac{\{\Theta_4\alpha J_{P_2-1}(\Theta_4\alpha) - (P+2)J_{P_2}(\Theta_4\alpha)\}}{\{\Theta_4\alpha X_{P_2-1}(\Theta_4\alpha) - (P+2)X_{P_2}(\Theta_4\alpha)\}} = \frac{\{\Theta_4\beta J_{P_2-1}(\Theta_4\beta) - (P+2)J_{P_2}(\Theta_4\beta)\}}{\{\Theta_4\beta X_{P_2-1}(\Theta_4\beta) - (P+2)X_{P_2}(\Theta_4\beta)\}} \tag{72}$$

Where,

$$P_2^2 = 3 - \frac{H^2\varsigma^2}{\chi}, \qquad \Theta_4^2 = \frac{2\rho_0\zeta^2 - \chi\varsigma^2}{\chi}.$$

Using Eq. (71) and Eq. (72), the phase velocity for this case is (same as above Eq. (45) and Eq. (46))

$$\frac{c_4^2}{c_{02}^2} = \left[\frac{\left(\frac{\eta_2}{2\pi}\right)^2}{\left(\frac{\alpha}{\lambda}\right)^2} + 1 \right] \tag{73}$$

Where, $c_{02}^2 = \chi / 2\rho_0$

6 Numerical analysis

The effect of non-homogeneity on torsional waves in an aeolotropic material made of viscoelastic solids has been studied. The numerical computation of phase velocity has been made for homogeneous and non-homogeneous pipe. The graphs are plotted for the two cases (l=0 and l=2). Different values of α/λ (diameter/wavelength) for homogeneous in the presence of magnetic field and non homogeneous case in the absence of magnetic field are calculated from Eq. (49) and Eq. (65) with the help of MATLAB. The variations elastic constants and presence of magnetic field in two curves have been obtained by choosing the following parameters for homogeneous and non-homogeneous aeolotropic pipe.

Table 1. Material properties

	l	Φ_0	α
Homogeneous Pipe	0	2.333	**10**
Inhomogeneous Pipe	**2**	**2.333**	**10**

The curves obtained in fig. 1 clearly show that the phase velocity for homogeneous as well as non-homogeneous case decreases inside the aeolotropic tube. The presence of magnetic field also reduces the speed of torsional waves in viscoelastic solids. These curves justify the results obtained in Eq. (50) and Eq. (52) mathematically given by Narain [3] and Chandrasekharaiahi [15]. We see that for homogeneous case when magnetic field is present and for non-homogeneous case when magnetic field is not present the variation i.e. shape of the curves is same. For non-homogeneous case, the elastic constants and the density of the tube are varying as the square of the radius vector.

Table 2. Material properties

	l	**H (Gauss)**	$\delta_{55} / \delta_{66}$
Homogeneous Pipe	0	0.32	**0.8**
Inhomogeneous Pipe	**2**	**0**	**0.8**

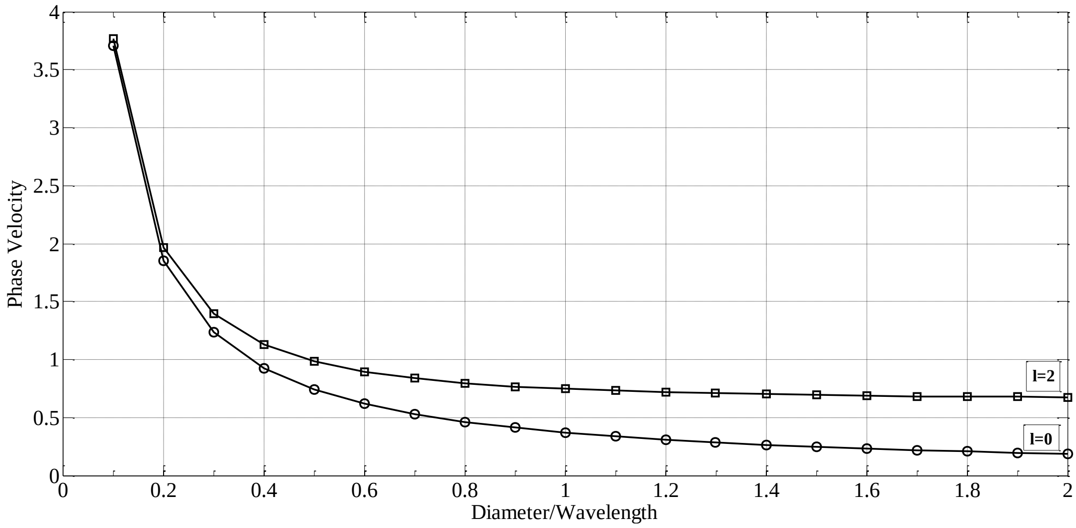

Fig.1 Torsional wave dispersion curves

7 Conclusion

The above problem deals with the interaction of elastic and electromagnetic fields in a viscoelastic media. This study is useful for detections of mechanical explosions inside the earth. In this study an attempt has been made to investigate the torsional wave propagation in non-homogeneous viscoelastic cylindrically aeolotropic material permeated by a magnetic field. It has been observed that the phase velocity decreases as the magnetic field increases.

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