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Derivation algebras of 3-Lie algebras G_i

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Abstract

The structure of derivation algebras and inner derivation algebras of 3-Lie algebras G_i (in Theorem 3.2) which are constructed by one dimensional extension of the complete Lie algebras L_i (in Lemma 2.2) is studied, $1 \leq i \leq 3$. It is proved that for complete Lie algebras L_i, we have $Der(G_i) \neq ad(G_i)$, and $\dim Der(G_i) = \dim ad(G_i) + 3$ for $i = 1, 2, 3$.

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1 Introduction

We know that 3-Lie algebras [1] have wide applications in mathematics and mathematical physics [2-3]. In the paper [4], authors constructed 3-Lie algebras by Lie algebras, which is a class of 3-Lie algebras $G = L \oplus Fx_0$ extended by Lie algebras L over a field F. It is proved that if subspace I of a Lie algebra L is an ideal of the 3-Lie algebra G if and only if I is an ideal of the Lie algebra L, and for an ideal I of the Lie algebra L, the subspace $J = I + Fx_0$ may not be an ideal of the 3-Lie algebra G, and the relationship between inner derivations of Lie algebra L and the inner derivations of the 3-Lie algebra G

is provided. In this paper, we continue to study the structure of the 3-Lie algebra G. We pay our main attention to the structure of derivation algebras. We discuss inner derivation algebras and derivation algebras of 3-Lie algebra G_i, $i=1,2,3$ which are realized by three complete Lie algebras L_i, $i=1,2,3$.

2 Derivation algebras of 3-Lie algebra G_i

In this section we discussion the derivation algebras of the 3-Lie algebras G_i which are constructed by the Lie algebras L_1, L_2 and L_3, respectively. First we introduce some basic notions.

An n-Lie algebra ($n \geq 2$) is a vector space A over a field F with an n-ary skew-symmetric multilinear multiplication $[,\cdots,]$ satisfying the following identities, for all $x_1,\cdots,x_n,\ y_2,\cdots,y_n \in A$,

$$[[x_1,\cdots,x_n],y_2,\cdots,y_n] = \sum_{i=1}^{n}[x_1,\cdots,[x_i,y_2,\cdots,y_n],\cdots,x_n].$$

If a Lie algebra L has trivial center and satisfies that all the derivations are inner derivations, then L is called *a complete Lie algebra.*

Lemma 2.1[4] Let $(L,[,])$ be a Lie algebra over a field F, x_0 be not contained in L. Set $G = L\dot{+}Fx_0$. Define the linear multiplication $[,,]: G\wedge G\wedge G \to G$ by $[x,y,x_0]=[x,y], [x,y,z]=0, \forall x,y,z\in L$. Then $(G,[,,])$ is a 3-Lie algebra.

Lemma 2,2 *The following Lie algebras are complete Lie algebras:*

L_1 : *L is a Lie algebra with a basis* $\{e_1,e_2\}$ *and the multiplication is* $[e_1,e_2]=e_1$.

L_2 : *L is a Lie algebra with a basis* $\{e_1,\cdots,e_8\}$ *and the multiplication is*

$[e_3,e_4]=e_5,\ [e_3,e_5]=e_6,\ [e_3,e_6]=e_7,\ [e_4,e_8]=e_7,\ [e_1,e_3]=e_3,$
$[e_1,e_5]=e_5,\ [e_1,e_6]=2e_6,\ [e_1,e_7]=3e_7,\ [e_1,e_8]=3e_8,\ [e_2,e_4]=e_4,$
$[e_2,e_5]=e_5,\ [e_2,e_6]=e_6,\ [e_2,e_7]=e_7.$

L_3 : *L is a Lie algebra with a basis* $\{e_1,\cdots,e_7\}$ *and the multiplication is*

$[e_1,e_3]=e_3,\ [e_1,e_5]=2e_5,\ [e_1,e_6]=e_6,\ [e_1,e_7]=2e_7,\ [e_2,e_4]=e_4,$
$[e_2,e_6]=e_6,\ [e_2,e_7]=e_7,\ [e_3,e_4]=e_6,\ [e_3,e_6]=e_7,\ [e_4,e_5]=e_7.$

Proof The result follows from the direct computation.

Let V be an m-dimensional vector space over a field F with a basis $\{v_1,\cdots,v_m\}$, $D: V\to V$ be a linear map, suppose $D(v_i)=\sum_{j=1}^{m}a_{ij}v_j$, $1\leq i\leq m$, then the matrix of D in the basis $\{v_1,\cdots,v_m\}$ is $A=(a_{ij})_{i,j=1}^{n}=\sum_{i,j=1}^{n}a_{ij}E_{ij}$, where E_{ij} are $(n\times n)$ matrix units, $1\leq i,j\leq n$.

Theorem 2.3 *Let* L_i, $i=1,2,3$ *be Lie algebras in Lemma 2.2. The derivation algebras of Lie algebras* L_i, $1\leq i\leq 3$, *are as follows:*

$L_1: Der(L_1)=ad(L_1)=FE_{11}+FE_{21}.$

$L_2: Der(L_2)=ad(L_2)=F(E_{33}+E_{55}+2E_{66}+3E_{77}+3E_{88})+F(E_{44}+E_{55}+E_{66}$
$+E_{77})+F(E_{13}-E_{45}-E_{56}-E_{67})+F(E_{24}+E_{35}-E_{87})+F(E_{15}+E_{25}+E_{36})$

$+F(E_{16}+\frac{1}{2}E_{26}+\frac{1}{2}E_{37})+F(E_{17}+\frac{1}{3}E_{27})+F(E_{18}+\frac{1}{3}E_{47})$.

$L_3: Der(L_3)=ad(L_3)=F(E_{33}+2E_{55}+E_{66}+2E_{77})+F(E_{44}+E_{66}+E_{77})$
$+F(E_{13}-E_{46}-E_{67})+F(E_{15}+\frac{1}{2}E_{47})+F(E_{16}+E_{26}+E_{37}+FE_{17}+FE_{27}$
$+F(E_{24}+E_{36}-E_{57})+FE_{83}+FE_{84}+FE_{85}+FE_{86}+FE_{87}$.

Proof The result follows from Lemma 2.2 and a direct computation.

Theorem 2.4 *Let L_i, $i=1,2,3$ be Lie algebras in Lemma 2.2, x_0 be not contained in L_i. Then the multiplication of* 3*-Lie algebras $G_i=L_i\dot{+}Fx_0$, are as following:* $G_1:[e_1,e_2,x_0]=e_1$.

$G_2:[e_3,e_4,x_0]=e_5,\ [e_3,e_5,x_0]=e_6,\ [e_3,e_6,x_0]=e_7,\ [e_4,e_8,x_0]=e_7,$
$[e_1,e_3,x_0]=e_3,\ [e_1,e_5,x_0]=e_5,\ [e_1,e_6,x_0]=2e_6,\ [e_1,e_7,x_0]=3e_7,$
$[e_1,e_8,x_0]=3e_8,\ [e_2,e_4,x_0]=e_4,\ [e_2,e_5,x_0]=e_5,\ [e_2,e_6,x_0]=e_6,$
$[e_2,e_7,x_0]=e_7,\ [e_i,e_j,e_k]=0,\ 1\le i,j,k\le 8.$

$G_3:[e_1,e_3,x_0]=e_3,\ [e_1,e_5,x_0]=2e_5,\ [e_1,e_6,x_0]=e_6,\ [e_1,e_7,x_0]=2e_7,$
$[e_2,e_4,x_0]=e_4,\ [e_2,e_6,x_0]=e_6,\ [e_2,e_7,x_0]=e_7,\ [e_3,e_4,x_0]=e_6,$
$[e_3,e_6,x_0]=e_7,\ [e_4,e_5,x_0]=e_7,\ [e_i,e_j,e_k]=0,\ 1\le i,j,k\le 7.$

Where $\{e_1,e_2,x_0\}$, $\{e_1,\cdots,e_8,x_0\}$ and $\{e_1,\cdots,e_7,x_0\}$ is the basis of the 3-Lie algebra G_1,G_2,G_3, respectively.

Proof The result follows from Lemma 2.1 and Lemma 2.2, directly.

Theorem 2.5 *Let G_i be* 3*-Lie algebras in Theorem 2.4, $i=1,2,3$. Then Derivation algebras and inner derivation algebras as follows*

$G_1: ad(G_1)=ad(L_1)+FE_{31},\ Der(G_1)=ad(G_1)+F(E_{22}-E_{33})+FE_{23}+FE_{32}$.

$G_2: ad(G_2)=ad(L_2)+FE_{93}+FE_{94}+FE_{95}+FE_{96}+FE_{97}+FE_{98}$,
$Der(G_2)=ad(G_2)+F(E_{11}+E_{22}-E_{55}-2E_{66}-3E_{77}-2E_{88}-E_{99})$
$+FE_{91}+FE_{92}$.

$G_3: ad(G_3)=ad(L_3)+FE_{83}+FE_{84}+FE_{85}+FE_{86}+FE_{87}$,
$Der(G_3)=ad(G_3)+F(E_{11}+E_{22}-E_{55}-E_{66}-2E_{77}-E_{88})+FE_{81}+FE_{82}$.

Proof Let d and D be an inner derivation and derivation of the 3-Lie algebra G_i, respectively. First we discuss the case G_1. Suppose the matrix of d and D in the basis $\{e_1,e_2,x_0\}$ is $\sum\limits_{i,j=1}^{3}a_{ij}E_{ij}$, where $a_{ij}\in F, 1\le i,j\le 3$. By Lemma 2.2 and the direct computation, the inner derivation $d=a_{11}E_{11}+a_{21}E_{21}+a_{31}E_{31}$, and the derivation $D=a_{11}E_{11}+a_{21}E_{21}+a_{31}E_{31}+a_{22}(E_{22}-E_{33}+a_{23}E_{23}+a_{32}E_{32})$.

For the 3-Lie algebra G_2, suppose the matrix form of inner derivation d and derivation D in the bases $\{e_1,\cdots,e_8,x_0\}$ is $\sum\limits_{i,j=1}^{9}a_{ij}E_{ij}$, where $a_{ij}\in F, 1\le i,j\le 9$. For derivation D, by the definition of derivations and Lemma 3.2,
$a_{12}=a_{14}=a_{16}=a_{19}=a_{21}=a_{23}=a_{26}=a_{28}=a_{29}=a_{31}=a_{32}=a_{34}=a_{37}$
$=a_{38}=a_{39}=a_{41}=a_{42}=a_{43}=a_{46}=a_{48}=a_{49}=a_{51}=a_{52}=a_{53}=a_{54}=$
$a_{57}=a_{58}=a_{59}=a_{61}=a_{62}=a_{63}=a_{64}=a_{65}=a_{68}=a_{69}=a_{71}=a_{72}=$
$a_{73}=a_{74}=a_{75}=a_{76}=a_{78}=a_{79}=a_{81}=a_{82}=a_{83}=a_{84}=a_{85}=a_{86}=$
$a_{89}=0,\ a_{17}=3a_{27},\ a_{18}=3a_{47},\ a_{13}=-a_{45}=-a_{56}=-a_{67},\ a_{15}=a_{25}=a_{36},$

$a_{24} = a_{35} = -a_{87}$, $a_{11} = a_{22} = -a_{99}$, $a_{55} = a_{33}+a_{44}-a_{11}$, $a_{66} = 2a_{33}+a_{44}-2a_{11}$, $a_{77} = 3a_{33} + a_{44} - 3a_{11}$, $a_{88} = 3a_{33} - 2a_{11}$. Therefore,

$D = a_{11}E_{11} + a_{13}E_{13} + a_{15}E_{15} + a_{16}E_{16} + a_{17}E_{17} + a_{18}E_{18} + a_{11}E_{22} + a_{24}E_{24} + a_{15}E_{25} + \frac{1}{2}a_{16}E_{26} + \frac{1}{3}a_{17}E_{27} + a_{33}E_{33} + a_{24}E_{35} + a_{15}E_{36} + \frac{1}{2}a_{16}E_{37} + a_{44}E_{44} - a_{13}E_{45} + \frac{1}{3}a_{18}E_{47} + (a_{33} + a_{44} - a_{11})E_{55} - a_{13}E_{16} + (2a_{33} + a_{44} - 2a_{11})E_{66} - a_{13}E_{67} + (3a_{33}+a_{44}-3a_{11})E_{77} - a_{24}E_{87} + (3a_{33}-2a_{11})E_{88} + \sum_{j=1}^{8} a_{9j}E_{9j} - a_{11}E_{99}$.

Similarly the inner derivation d has the form
$d = a_{13}E_{13} + a_{15}E_{15} + a_{16}E_{16} + a_{17}E_{17} + a_{18}E_{18} + a_{24}E_{24} + a_{15}E_{25} + \frac{1}{2}a_{16}E_{26} + \frac{1}{3}a_{17}E_{27} + a_{33}E_{33} + a_{24}E_{35} + a_{15}E_{36} + \frac{1}{2}a_{16}E_{37}\ a_{44}E_{44} - a_{13}E_{45} + \frac{1}{3}a_{18}E_{47} + (a_{33} + a_{44})E_{55} - a_{13}E_{56} + (2a_{33} + a_{44})E_{66} - a_{13}E_{67} + (3a_{33} + a_{44})E_{77} - a_{24}E_{87} + 3a_{33}E_{88} + a_{93}E_{93} + a_{94}E_{94} + a_{95}E_{95} + a_{96}E_{96} + a_{97}E_{97} + a_{98}E_{98}$.

For the case G_3, by the completely similar discussion we have
$d = a_{13}E_{13} + a_{15}E_{15} + a_{16}E_{16} + a_{17}E_{17} + a_{24}E_{24} + a_{16}E_{26} + \frac{1}{2}a_{17}E_{27} + a_{33}E_{33} + a_{24}E_{36} + a_{16}E_{37} + a_{44}E_{44} - a_{13}E_{46} + \frac{1}{2}a_{15}E_{47} + 2a_{33}E_{54} - a_{24}E_{56}\ (a_{33} + a_{44})E_{66} - a_{13}E_{67} + (2a_{33} + a_{44})E_{77} + a_{83}E_{83} + a_{84}E_{84} + a_{85}E_{85} + a_{86}E_{86} + a_{87}E_{87}$.
$D = a_{11}E_{11} + a_{13}E_{13} + a_{15}E_{15} + a_{16}E_{16} + a_{17}E_{17} + a_{11}E_{22} + a_{24}E_{24} + a_{16}E_{26} + \frac{1}{2}a_{17}E_{26} + a_{33}E_{33} + a_{24}E_{36} + a_{16}E_{37} + a_{44}E_{44} - a_{13}E_{46} + \frac{1}{2}a_{15}E_{47} - a_{24}E_{57} + (2a_{33} - a_{11})E_{55} + (a_{33} + a_{44} - a_{11})E_{66} - a_{13}E_{67} + (2a_{33} + a_{44} - 2a_{11})E_{77} + \sum_{j=1}^{7} a_{8j}E_{8j} - a_{11}E_{88}$. Thanks to Theorem 3.1, the result follows.

Theorem 2.6 *Let G_i be 3-Lie algebras in Theorem 2.4, $i = 1, 2, 3$. Then $Der(G_i) \neq ad(G_i)$, and $\dim Der(G_i) = \dim ad(G_i) + 3$, $i = 1, 2, 3$.*

Proof The result follows from Theorem 2.5, directly.

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