Coclosed-Exact Fields of Differential Forms

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Abstract

In the present paper, we first give the definition for coclosed-exact fields of differential forms, and then an estimate below the natural exponents of coclosed-exact forms is obtained. An application to the regularity theory of quasiregular mappings is given.

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1 Introduction

We first introduce some basic notions of exterior calculus. Throughout this paper we always assume Ω is a connected open subset of \( \mathbb{R}^n \), \( n \geq 2 \). We use \( e_1, e_2, \ldots, e_n \) to denote the standard unit basis of \( \mathbb{R}^n \). Let \( \Lambda^\ell = \Lambda^\ell(\mathbb{R}^n) \) be the linear space of \( \ell \)-covectors, spanned by the exterior products \( e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell} \), corresponding to all ordered \( \ell \)-tuples \( I = (i_1, i_2, \cdots, i_\ell) \), \( 1 \leq i_1 < i_2 < \cdots < i_\ell \leq n \), \( \ell = 0, 1, \cdots, n \). The Grassman algebra \( \Lambda = \bigoplus \Lambda^\ell \) is a graded algebra with respect to the exterior products. For \( \alpha = \sum \alpha_I e_I \in \Lambda \) and \( \beta = \sum \beta_I e_I \in \Lambda \), the inner product in \( \Lambda \) is given by \( \langle \alpha, \beta \rangle = \sum \alpha_I \beta_I \) with summation over all \( \ell \)-tuples \( I = (i_1, i_2, \cdots, i_\ell) \) and all integers \( \ell = 0, 1, \cdots, n \). The Hodge star operator \( \ast : \Lambda^\ell \to \Lambda^{n-\ell} \) is defined by the rule \( \ast 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \) and \( \alpha \wedge \ast \beta = \beta \wedge \ast \alpha = \langle \alpha, \beta \rangle(\ast 1) \) for all \( \alpha, \beta \in \Lambda \). The norm of \( \alpha \in \Lambda \) is given by the formula \( |\alpha|^2 = \langle \alpha, \alpha \rangle = \ast (\alpha \wedge \ast \alpha) \in \Lambda^0 = \mathbb{R} \). The Hodge star is an isometric isomorphism on \( \Lambda \) with \( \ast : \Lambda^\ell \to \Lambda^{n-\ell} \) and \( \ast \ast = (-1)^{\ell(n-\ell)} : \Lambda^\ell \to \Lambda^\ell \).

Let \( \mathcal{D}'(\Omega, \Lambda^\ell) \) be those differential forms \( \omega = \sum \omega_I e_I \in \Lambda^\ell \) with \( \omega_I \in \mathcal{D}'(\Omega) \), where we have denoted by \( \mathcal{D}'(\Omega) \) the space of Schwartz distributions. Let \( 1 \leq p < \infty \). We denote the \( L^p \)-norm of a measurable function \( f \) over \( \Omega \) by

\[
\|f\|_p = \|f\|_{p,\Omega} = \left( \int_{\Omega} |f(x)|^p dx \right)^{1/p}.
\]
We write $L^p \left( \Omega, \bigwedge^\ell \right)$ for the $\ell$-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 \ldots i_n} (x) dx_{i_1} \wedge \ldots \wedge dx_{i_\ell}$ with $\omega_I(x) \in L^p(\Omega, \mathbb{R})$ for all ordered $\ell$-tuples $I$. Thus $L^p \left( \Omega, \bigwedge^\ell \right)$ is a Banach space with norm

$$
\|\omega\|_{p, \Omega} = \left( \int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left( \int_\Omega \left( \sum |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.
$$

Similarly, $W^{1,p} \left( \Omega, \bigwedge^\ell \right)$ are those differential $\ell$-forms on $\Omega$ whose coefficients are in $W^{1,p}(\Omega, \mathbb{R})$. The notations $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R})$ and $W^{1,p}_{\text{loc}} \left( \Omega, \bigwedge^\ell \right)$ are self-explanatory.

The exterior derivative is denoted by $d: \mathcal{D}' \left( \Omega, \bigwedge^\ell \right) \to \mathcal{D}' \left( \Omega, \bigwedge^{\ell+1} \right)$ for $\ell = 0, 1, \ldots, n$. Its formal adjoint operator $d^*: \mathcal{D}' \left( \Omega, \bigwedge^{\ell+1} \right) \to \mathcal{D}' \left( \Omega, \bigwedge^\ell \right)$ is given by $d^* = (-1)^{n\ell+1} \ast d^\ast$ on $\mathcal{D}' \left( \Omega, \bigwedge^{\ell+1} \right)$, $\ell = 0, 1, \ldots, n$. The well-known Poincaré Lemma states that $d \circ d = 0$. It is easy to see that $d^* \circ d^* = 0$ as well.

A differential $\ell$-form $u \in \mathcal{D}' \left( \Omega, \bigwedge^\ell \right)$ is called a closed form if $du = 0$ in $\Omega$. It is called exact if there exists a differential form $\alpha \in \mathcal{D}' \left( \Omega, \bigwedge^{\ell+1} \right)$ such that $u = d\alpha$. Poincaré Lemma implies that exact forms are closed. Similarly, a differential $\ell$-form $v \in \mathcal{D}' \left( \Omega, \bigwedge^\ell \right)$ is called a coclosed form if $d^*v = 0$. It is called coexact if there exists a differential $(\ell + 1)$-form $\beta \in \mathcal{D}' \left( \Omega, \bigwedge^{\ell+1} \right)$ such that $v = d^*\beta$. Poincaré Lemma implies that coexact forms are coclosed.

Let $G = \left( G^L_{i_j} \right)_{1 \leq i, j \leq n}$ be an $n \times n$ matrix. The $\ell$-exterior power of $G$ is a linear operator

$$
G^\ell_{\#} : \bigwedge^\ell \to \bigwedge^\ell
$$

defined by

$$
G^\ell_{\#}(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_\ell) = G\alpha_1 \wedge G\alpha_2 \wedge \cdots \wedge G\alpha_\ell,
$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell \in \bigwedge^1$. The linear transform $G^\ell_{\#}$ can be expressed as an $G^\ell_n \times G^\ell_n$ matrix whose entries are $\ell \times \ell$ minors of $G$ and denoted by $G^\ell_{\#} = (\det G^L_J)_{G^\ell_n \times G^\ell_n}$, where $I = (i_1, \ldots, i_\ell)$, $J = (j_1, \ldots, j_\ell)$ are ordered $\ell$-tuples and

$$
\det G^L_J = \det \begin{bmatrix} G^L_{i_1, j_1} & \cdots & G^L_{i_\ell, j_1} \\ \vdots & \ddots & \vdots \\ G^L_{i_1, j_\ell} & \cdots & G^L_{i_\ell, j_\ell} \end{bmatrix}.
$$

**Definition 1.1** A pair of differential $\ell$-forms $\mathcal{F} = (\mathcal{C}, \mathcal{E}) \in L^{p'} \left( \Omega, \bigwedge^{\ell-1} \right) \times L^{q'} \left( \Omega, \bigwedge^\ell \right)$, $1 \leq p', q' < \infty$, is called coclosed-exact, if $d^* \mathcal{C} = 0$ and there exists a differential $(\ell - 1)$-form $u \in \bigwedge^{\ell-1}$ such that $\mathcal{E} = du$. Moreover, the Jacobian associated to the field $\mathcal{F}$ is defined by $J(x, \mathcal{F}) = \langle \mathcal{C}, \mathcal{E} \rangle$.

In much the same way, we can define the closed-coexact fields of differential forms.
Definition 1.2 A pair of differential $\ell$-forms $\mathcal{F} = (C, E) \in L^{p'} \left( \Omega, \Lambda^\ell \right) \times L^{q'} \left( \Omega, \Lambda^{\ell+1} \right)$, $1 \leq p', q' < \infty$, is called closed-coexact, if $dC = 0$ and there exists a differential $(\ell+1)$-form $u \in \Lambda^{\ell+1}$ such that $E = d^*u$. The Jacobian associated to the field $\mathcal{F}$ is defined by $J(x, \mathcal{F}) = (C, E)$.

Balls with radius $R$ are denoted by $B_R$ and $B_{\sigma R}$ is the ball with the same center as $B_R$ and $\text{diam}(B_{\sigma R}) = \sigma \text{diam}(B_R)$. The $n$-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$. We can find the following result in [1, 2]: Let $Q \subset \mathbb{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty \left( Q, \Lambda^\ell \right) \to C^\infty \left( Q, \Lambda^{\ell-1} \right)$ defined by

$$(K_y \omega)(x; \xi_1, \xi_2, \cdots, \xi_{\ell-1}) = \int_0^1 t^{\ell-1} \omega(tx + ty - ty; x - y, \xi_1, \cdots, \xi_{\ell-1}) dt$$

and the decomposition

$$\omega = d(K_y) + K_y(d\omega).$$

Another linear operator $T_Q : C^\infty \left( Q, \Lambda^\ell \right) \to C^\infty \left( Q, \Lambda^{\ell-1} \right)$ is defined by averaging $K_y$ over all points $y$ in $Q$

$$T_Q \omega = \int_Q \varphi(y) K_y \omega dy,$$

where $\varphi \in C^\infty_0(Q)$ is normalized by $\int_Q \varphi(y) dy = 1$. We define the $\ell$-form $\omega_Q \in D' \left( Q, \Lambda^\ell \right)$ by

$$\omega_Q = |Q|^{-1} \int_Q \omega(y) dy, \quad \text{if } \ell = 0, \quad \text{and } \omega_Q = d(T_Q \omega), \quad \text{if } \ell = 1, 2, \cdots, n,$$

for all $\omega \in L^p \left( Q, \Lambda^\ell \right), 1 \leq p < \infty$. It is easy to see that $\omega_Q$ is exact.

2 Estimates Below the Natural Exponents

In this section, we derive two estimates below the natural exponents for coclosed-exact and closed-coexact fields of differential forms.

In the following, we denote by $c(\ast, \cdots, \ast)$ a constant depending only on the quantities $\ast, \cdots, \ast$, whose value may be different from line to line.

We begin with a simple consequence of Hölder’s inequality. Let $1 < p', q' < \infty$ be a Hölder conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$. For any pair of differential forms $\mathcal{F} = (C, E)$ with $C \in L^{p'} \left( B_R, \Lambda^\ell \right)$, $E \in L^{q'} \left( B_R, \Lambda^{\ell+1} \right)$, and any test function $\varphi \in C^\infty_0(B_R)$, we have

$$\left| \int_{B_R} \varphi J(x, \mathcal{F}) dx \right| = \left| \int_{B_R} \varphi \langle C, E \rangle dx \right| \leq \|\varphi\|_\infty \|C\|_{p'} \|E\|_{q'}.$$
In order to exploit certain cancelations in the above integral we now assume $F = (C, E)$ be a coclosed-exact pair of differential forms, that is, $d^* C = 0$ and there exists a differential $(\ell - 1)$-form $u \in \Lambda^{\ell-1}$ such that $E = du$. Unless otherwise stated, this assumption will remain valid throughout this article. The following theorems are estimates with integrable exponents below the natural ones. For coclosed-exact fields of differential forms, we have

**Theorem 2.1** Let $1 < p', q' < \infty$ be a H"older conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$, and $1 < r', s' < \infty$ satisfies $\frac{1}{p'} + \frac{1}{s'} = 1 + \frac{1 + \varepsilon}{n}$. Then there exists a constant $c = c(n, p', r')$ such that for each test function $\psi \in C_0^\infty(B_R)$, one has

$$
\int_{B_R} \psi^{1-\varepsilon} J(x, F) \frac{dx}{|C| |E|^\varepsilon} \leq c \varepsilon \|C\|_{L_{p'}^{1-\varepsilon}(E)} \|d(\psi(u-u_{B_R}))\|_{L_{q'}^{1-\varepsilon}(E)} + c \|\nabla \psi\|_{L_{s'}^{1-\varepsilon}(E)} \|du\|_{L_{r'}^{1-\varepsilon}(E)},
$$

(2.1)

wherever $0 \leq 2\varepsilon \leq \min \left\{ \frac{q'-1}{q'}, \frac{q'-1}{q'}, \frac{q'-1}{r'}, \frac{q'-1}{s'} \right\}$ and $F = (C, E) \in L^{p'}(B_R, \Lambda^\ell) \times L^{q'}(B_R, \Lambda^\ell) \cap L^{r'}(B_R, \Lambda^\ell) \times L^{s'}(B_R, \Lambda^\ell)$ a coclosed-exact field of differential $\ell$-forms.

For closed-coexact fields of differential forms, we have

**Theorem 2.2** Let $1 < p', q' < \infty$ be a H"older conjugate pair, $\frac{1}{p'} + \frac{1}{q'} = 1$, and $1 < r', s' < \infty$ satisfies $\frac{1}{p'} + \frac{1}{s'} = 1 + \frac{1 + \varepsilon}{n}$. Then there exists a constant $c = c(n, p', r')$ such that (2.1) holds for each test function $\varphi \in C_0^\infty(B_R)$, wherever $0 \leq 2\varepsilon \leq \min \left\{ \frac{q'-1}{q'}, \frac{q'-1}{q'}, \frac{q'-1}{r'}, \frac{q'-1}{s'} \right\}$ and $F = (C, E) \in L^{p'}(B_R, \Lambda^\ell) \times L^{q'}(B_R, \Lambda^\ell) \cap L^{r'}(B_R, \Lambda^\ell) \times L^{s'}(B_R, \Lambda^\ell)$ a closed-coexact field of differential $\ell$-forms.

The key tool used in establishing (2.1) is the stability of the Hodge decomposition theorem under nonlinear perturbations of differential forms, first discovered by Iwaniec [3].

**Lemma 2.3** For $\omega \in L^{r'(1-\varepsilon)}(\Lambda^\ell, \varepsilon < \frac{1}{2})$, consider the Hodge decomposition

$$
|\omega|^{-\varepsilon} \omega = d\alpha + d^* \beta, \quad \text{with } \alpha \in L_{1}^{r'(1-\varepsilon)}(\Lambda^{\ell-1}) \quad \text{and } \beta \in L_{1}^{r'(1-\varepsilon)}(\Lambda^{\ell+1}).
$$

If $\omega$ is closed, then

$$
\|d^* \beta\|_{r} \leq c(n) r |\varepsilon| \|\omega\|_{r'(1-\varepsilon)}^{1-\varepsilon}.
$$

If $\omega$ is coclosed, then

$$
\|d\alpha\|_{r} \leq c(n) r |\varepsilon| \|\omega\|_{r'(1-\varepsilon)}^{1-\varepsilon}.
$$
In the proof of Theorem 2.1, we will also need the Poincaré and Sobolev-
Poincaré inequalities, which can be found in [2], see also [4, 5].

**Lemma 2.4** Suppose that $\omega \in \mathcal{D}'(B, \Lambda^{\ell})$ and $d\omega \in L^p(B, \Lambda^{\ell+1})$, $\ell = 0, 1, \cdots, n$. Then $\omega - \omega_B$ is in $L^p(B, \Lambda^{\ell})$ and we have the following uniform estimate

$$
\left( \int_B |\omega - \omega_B|^p dx \right)^{1/p} \leq C(p, n) \text{diam}(B) \left( \int_B |d\omega|^p dx \right)^{1/p}
$$

for $B$ a cube or a ball in $\mathbb{R}^n$.

**Lemma 2.5** Suppose that $\omega \in \mathcal{D}'(B, \Lambda^{\ell})$ and $d\omega \in L^p(B, \Lambda^{\ell+1})$, $\ell = 0, 1, \cdots, n$ and $1 < p < n$. Then $\omega - \omega_B$ is in $L^{np/(n-p)}(B, \Lambda^{\ell})$ and we have the following uniform estimate

$$
\left( \int_B |\omega - \omega_B|^{np/(n-p)} dx \right)^{(n-p)/np} \leq C(p, n) \left( \int_B |d\omega|^p dx \right)^{1/p}
$$

(2.2)

for $B$ a cube or a ball in $\mathbb{R}^n$.

The following lemma comes from [6], which is an elementary inequality for
differential $\ell$-forms.

**Lemma 2.6** Suppose that $X, Y \in \Lambda^{\ell}$ be two differential $\ell$-forms and $0 \leq \varepsilon < 1$. Then

$$
|||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y|| \leq \frac{2\varepsilon(1 + \varepsilon)}{1 - \varepsilon} |X - Y|^{1-\varepsilon}.
$$

**Proof of Theorem 2.1** We define the values of the coefficients of $C$ and $\mathcal{E}$ to be 0 outside $B_R$. Let us decompose, according to Lemma 2.3, with $\omega = C \in L^{p/(1-\varepsilon)}(B_R, \Lambda^{\ell})$,

\[
\left\{ \begin{array}{c}
|C|^{-\varepsilon}C = d\alpha_1 + d^*\beta_1, \\
\alpha_1 \in L^{1}_{1}(B_R, \Lambda^{\ell-1}), \\
\beta_1 \in L^{q}_{1}(B_R, \Lambda^{\ell+1})
\end{array} \right., \\
\|d\alpha_1\|_{p'} \leq c(n)p'|\varepsilon||C|^{1-\varepsilon}_{p'(1-\varepsilon)},
\]

(2.3)

and then with $\omega = d(\psi(u - u_{BR})) \in L^{p/(1-\varepsilon)}(B_R, \Lambda^{\ell})$,

\[
\left\{ \begin{array}{c}
|d(\psi(u - u_{BR}))|^{-\varepsilon}d(\psi(u - u_{BR})) = d\beta_2, \\
\alpha_2 \in L^{1}_{q}(B_R, \Lambda^{\ell-1}), \\
\beta_2 \in L^{q}_{q}(B_R, \Lambda^{\ell+1})
\end{array} \right., \\
\|d^*\beta_2\|_{q'} \leq c(n)q'|\varepsilon||d(\psi(u - u_{BR}))|^{1-\varepsilon}_{q'(1-\varepsilon)}.
\]

(2.4)

(2.3) and (2.4) imply

$$
\|d^*\beta_1\|_{p'} \leq c(n)p'|\varepsilon||C|^{1-\varepsilon}_{p'(1-\varepsilon)}
$$

(2.5)
and
\[ \|d\alpha_2\|_{q'} \leq c(n)q' \|d(\psi(u - u_{BR}))\|_{q'(1-\varepsilon)}^{1-\varepsilon} \] (2.6)
respectively.

Let us introduce a differential \( \ell \)-form
\[ E = |d(\psi(u - u_{BR}))|^{-\varepsilon} d(\psi(u - u_{BR})) - |\psi du|^{-\varepsilon} \psi du, \]
then by Lemma 2.5 one has
\[ |E| \leq \frac{2^\varepsilon(1 + \varepsilon)}{1 - \varepsilon} |d\psi \wedge (u - u_{BR})|^{1-\varepsilon}. \] (2.7)
Since coclosed forms are orthogonal to exact forms, then
\[
\begin{align*}
|I_1| &= \left| \int_{B_R} \langle \alpha_1, \alpha_2 \rangle \right| \leq \|\alpha_1\|_{p'} \|\alpha_2\|_{q'} \\
&\leq c(n, p') \varepsilon \|\|C\|_{p'(1-\varepsilon)} \|d(\psi(u - u_{BR}))\|_{q'(1-\varepsilon)}^{1-\varepsilon}.
\end{align*}
\] (2.9)

\( |I_2| \) can be estimated by (2.4) and (2.6) as
\[
\begin{align*}
|I_2| &= \left| \int_{B_R} \langle d\beta_1, d\beta_2 \rangle \right| \leq \|d\beta_1\|_{p'} \|d\beta_2\|_{q'} \\
&\leq c(n, p') \varepsilon \|\|C\|_{p'(1-\varepsilon)} \|d(\psi(u - u_{BR}))\|_{q'(1-\varepsilon)}^{1-\varepsilon}.
\end{align*}
\] (2.10)

\( |I_3| \) can be estimated by (2.7) and Lemma 2.4 as
\[
\begin{align*}
|I_3| &= \left| \int_{B_R} \langle |C|^{-\varepsilon} C, E \rangle \right| dx \\
&\leq \frac{2^\varepsilon(1 + \varepsilon)}{1 - \varepsilon} \int_{B_R} |C|^{-\varepsilon} |d\psi \wedge (u - u_{BR})|^{1-\varepsilon} dx \\
&\leq c(n) \|\nabla \psi\|_{1-\varepsilon}^{-1} \int_{B_R} |C|^{-\varepsilon} |u - u_{BR}|^{1-\varepsilon} dx \\
&\leq c(n) \|\nabla \psi\|_{1-\varepsilon}^{-1} \left( \int_{B_R} |C|^{1-\varepsilon} dx \right)^{1/p'} \left( \int_{B_R} |u - u_{BR}|^{(1-\varepsilon)/(p'-1)} dx \right)^{(p'-1)/p'} \\
&\leq c(n, p') \|\nabla \psi\|_{1-\varepsilon}^{-1} \|C\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|du\|_{s'(1-\varepsilon)}^{1-\varepsilon},
\end{align*}
\] (2.11)
where we recall that \( \frac{1}{r} + \frac{1}{s} = 1 + \frac{1-\varepsilon}{n} \). Combining (2.8)-(2.11) we arrive at (2.1), completing the proof of Theorem 2.1.

**Proof of Theorem 2.2** Similar to the proof of Theorem 2.1.

### 3 An Application to Weakly Quasiregular Mappings

We now give an application of Theorem 2.1 to quasiregular mappings. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) and \( f = (f^1, f^2, \cdots, f^n) \in W^{1,r}_{loc}(\Omega, \mathbb{R}^n), 1 \leq r < \infty. \) The differential \( Df(x) : \Omega \rightarrow GL(n) \) and its determinant \( J_f(x) = \det Df(x) \) are, therefore, defined almost everywhere in \( \Omega \). We assume that \( J_f(x) \) is nonnegative.

**Definition 3.1** A mapping \( f \in W^{1,r}_{loc}(\Omega, \mathbb{R}^n) \) is said to be weakly \( K \)-quasiregular, \( 1 \leq K < \infty \), if

\[
\max_{|\xi|=1} |Df(x)\xi| \leq K \min_{|\xi|=1} |Df(x)\xi|
\]

for almost every \( x \in \Omega. \) It is called \( K \)-quasiregular if \( r \) is equal to the dimension of the domain, thus \( J_f(x) \in L^1_{loc}(\Omega). \)

The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations; see [7, 8] and the references therein. For the recent developments of quasiregular mapping theory, see [7-12].

If we introduce, for every \( K \)-quasiregular mapping \( f \), a metric tensor \( G(x) \) on \( \Omega, \)

\[
G(x) = \begin{cases} 
J_f^{-2/n}(x)D^tf(x)Df(x), & \text{for } J_f(x) \neq 0, \\
\text{Id}, & \text{for } J_f(x) = 0,
\end{cases}
\]

where \( D^tf(x) \) and \( \text{Id} \) are the transpose of \( Df(x) \) and the identity matrix, respectively, then quasiregular mappings are simply weak solutions to the differential system

\[
D^tf(x)Df(x) = J_f^{2/n}(x)G(x),
\]

commonly called the \( n \)-dimensional Beltrami equation.

Fix an ordered \( \ell \)-tuple \( I = (i_1, i_2, \cdots, i_\ell) \) and its complementary \( (n-1) \)-tuple \( J = (j_1, j_2, \cdots, j_{n-\ell}) \) ordered in such a way that \( dx_I = *dx_J \). Suppose that \( r \geq \max\{\ell, n-\ell\} \). To each such pair \((I, J)\) we assign the differential form

\[
u_I = f^{i_1}df^{i_1} \wedge \cdots \wedge df^{i_{\ell-1}} \in L_{loc}^{n/(n-1)}(\Omega, \wedge^{\ell-1})
\]
and the conjugate form
\[ v_J = * f^{j_1} df^{j_2} \wedge \cdots \wedge df^{j_{n-\ell}} \in L_{loc}^{n/(n-1)} \left( \Omega, \bigwedge^{\ell+1} \right). \]

The degree of local integrability is verified by the Sobolev embedding theorem. Clearly,
\[ du_I = (-1)^{\ell-1} df^{i_1} \wedge \cdots \wedge df^{i_{\ell}} \in L_{loc}^1 \left( \Omega, \bigwedge^{\ell} \right) \]
and
\[ d^* v_J = (-1)^{\ell+1} * df^{j_1} \wedge \cdots \wedge df^{j_n-\ell} \in L_{loc}^1 \left( \Omega, \bigwedge^{\ell} \right). \]

From [3], we know that the differential forms \( du_I, d^* v_J \in L_{loc}^1(\Omega, \bigwedge^{\ell}) \) satisfy the \( p \)-harmonic and the conjugate \( q \)-harmonic equations
\[
\begin{align*}
    d^* A(x, du_I) &= 0 \quad (3.1) \\
    dA^{-1}(x, d^* v_J) &= 0 \quad (3.2)
\end{align*}
\]
respectively, where
\[
\begin{align*}
    A(x, \xi) &= \langle (G^\ell_\#)^{-1}(x)\xi, \xi \rangle^{(p-2)/2}(G^\ell_\#)^{-1}(x)\xi, \quad p = \frac{n}{\ell}, \\
    A^{-1}(x, \xi) &= \langle (G^\ell_\#)(x)\xi, \xi \rangle^{(q-2)/2}(G^\ell_\#)(x)\xi, \quad q = \frac{n}{n-\ell},
\end{align*}
\]
and the following estimates hold
\[
\begin{align*}
    \langle A(x, du_I), du_I \rangle &\geq c_1 |du_I|^p, \quad (3.4) \\
    |A(x, du_I)| &\leq c_2 |du_I|^{p-1}. \quad (3.5)
\end{align*}
\]

We now give an alternative proof of Theorem 3.1 by using Theorem 2.1. Similarly, Theorem 3.1 can also be proved by using Theorem 2.2.

We recall a famous regularity result due to T. Iwaniec, see [3, Theorem 3].

**Theorem 3.2** There exist exponents \( q = q(n, K) < n < p(n, K) = p \) such that every weakly \( K \)-quasiregular mapping of class \( W_{loc}^{1,q}(\Omega, R^n) \) belongs to \( W_{loc}^{1,p}(\Omega, R^n) \) and so is \( K \)-quasiregular.
Lemma 3.3 For every weakly K-quasiregular mapping of class $W^{1,n(1-\varepsilon)}_{loc}(\Omega, \mathbb{R}^n)$, we have the weakly reverse Hölder inequality
\[
\int_{B_{R/2}} |du_I|^{p(1-\varepsilon)} dx \leq \theta \int_{B_R} |du_I|^{p(1-\varepsilon)} dx + \left( \int_{B_R} |du_I|^{np(1-\varepsilon)} dx \right)^{\frac{n+1-\varepsilon}{n}}.
\] (3.6)
provided that $\varepsilon$ small enough, where $f_{B_R} = \frac{1}{|B_R|} \int_B$ is the integral mean over $B_R$.

Proof. For quasiregular mapping $f \in W^{1,n(1-\varepsilon)}_{loc}(\Omega, \mathbb{R}^n)$, we introduce two differential 1-forms $C = A(x, du_I)$ and $\mathcal{E} = du_I$, then by (3.1), it is obvious that $\mathcal{F} = (C, \mathcal{E})$ is a coclosed-exact pair. For $B_R \subset \subset \Omega$, take $\psi \in C^\infty_0(B_R)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $B_{R/2}$ and $|\nabla \psi| \leq \frac{c(n)}{R}$. Then by (3.4) and (3.5),
\[
\int_{B_R} \psi^{1-\varepsilon} \mathcal{F}(x, \mathcal{F}) dx = \int_{B_R} \psi^{1-\varepsilon} \frac{\langle A(x, du_I), du_I \rangle}{|A(x, du_I)|^\varepsilon} dx \geq c \int_{B_{R/2}} |du_I|^{p(1-\varepsilon)} dx.
\] (3.7)
Take $p' = \frac{p}{p-1}$ and $q' = p$ we obtain from Lemma 2.4 that
\[
\varepsilon \|C\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|du_I - (u_I)_{B_R}\|_{q'(1-\varepsilon)}^{1-\varepsilon} \\
\leq \varepsilon \|C\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|\psi du_I\|_{q'(1-\varepsilon)}^{1-\varepsilon} + \|\psi \vee (u_I - (u_I)_{B_R})\|_{q'(1-\varepsilon)}^{1-\varepsilon} \\
\leq c \varepsilon \|C\|_{p'(1-\varepsilon)}^{1-\varepsilon} \left[ \|du_I\|_{q'(1-\varepsilon)}^{1-\varepsilon} - \frac{1}{R^{1-\varepsilon}} \|(u_I - (u_I)_{B_R})\|_{q'(1-\varepsilon)}^{1-\varepsilon} \right] \\
\leq c \varepsilon \|du_I\|_{p'(1-\varepsilon)}^{1-\varepsilon} \left[ \|du_I\|_{q'(1-\varepsilon)}^{1-\varepsilon} + \frac{1}{R^{1-\varepsilon}} \|(u_I - (u_I)_{B_R})\|_{q'(1-\varepsilon)}^{1-\varepsilon} \right] \\
\leq c \varepsilon \|du_I\|_{p'(1-\varepsilon)}^{1-\varepsilon}.
\] (3.8)
Take $r' = \frac{n}{(p-1)(n+1-\varepsilon)}$ and $s' = \frac{np}{n+1-\varepsilon}$, we obtain
\[
\|\nabla \psi\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|C\|_{r'(1-\varepsilon)}^{1-\varepsilon} \|du_I\|_{s'(1-\varepsilon)}^{1-\varepsilon} \\
\leq \frac{c}{R^{1-\varepsilon}} \|du_I\|_{p'(1-\varepsilon)}^{1-\varepsilon} \|du_I\|_{np'(1-\varepsilon)}^{1-\varepsilon} \\
= \frac{c}{R^{1-\varepsilon}} \|du_I\|_{np'(1-\varepsilon)}^{1-\varepsilon}.
\] (3.9)
Combining (2.1) with (3.7), (3.8) and (3.9) we get that
\[
\int_{B_{R/2}} |du_I|^{p(1-\varepsilon)} dx \leq c \varepsilon \int_{B_R} |du_I|^{p(1-\varepsilon)} dx + \frac{c}{R^{1-\varepsilon}} \left( \int_{B_R} |du_I|^{np(1-\varepsilon)} dx \right)^{\frac{n+1-\varepsilon}{n}}.
\]
Divide both sides of the above inequality by $|B_{R/2}| = \omega_n(R/2)^n$ we obtain
\[
\frac{1}{|B_{R/2}|} \int_{B_{R/2}} |du_I|^{p(1-\varepsilon)} dx \leq c \varepsilon \int_{B_R} |du_I|^{p(1-\varepsilon)} dx + \frac{c}{R^{1-\varepsilon}} \left( \int_{B_R} |du_I|^{np(1-\varepsilon)} dx \right)^{\frac{n+1-\varepsilon}{n}}.
\]
Take $\varepsilon$ small enough such that $\theta = c \varepsilon < 1$, we arrive at (3.7). Lemma 3.3 has been proved.

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References


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