In this paper, we present the concepts of \((\in, \in \lor q_k)\)-fuzzy ideals in ternary semigroups, which is a generalization of the \((\in, \in \lor q)\) fuzzy ideals of a ternary semigroups. In this regard, we define \((\in, \in \lor q_k)\)-fuzzy left (right, lateral) ideals, \((\in, \in \lor q_k)\)-fuzzy quasi-ideals and \((\in, \in \lor q_k)\)-fuzzy bi-ideals and prove some basic results using these definitions. Special concentration is paid to \((\in, \in \lor q_k)\)-fuzzy left (right, lateral) ideals, \((\in, \in \lor q_k)\)-fuzzy quasi-ideals and \((\in, \in \lor q_k)\)-fuzzy bi-ideals. Furthermore, we characterize regular ternary semigroups in terms of these notions.

**Mathematics Subject Classification:** 06D72, 20N10, 20M12

**Keywords:** Fuzzy subsets, \((\in, \in \lor q_k)\)-fuzzy left (right, lateral) ideals, \((\in, \in \lor q_k)\)-fuzzy quasi-ideals, \((\in, \in \lor q_k)\)-fuzzy bi-ideals, Regular ternary semigroups
1 Introduction

In 1932, Lehmer introduced the concept of a ternary semigroups [22]. A nonempty set $X$ is called a ternary semigroup if there exists a ternary operation $X \times X \times X \rightarrow X$; written as $(x_1, x_2, x_3) \rightarrow x_1 x_2 x_3$ satisfying the following identity for any $x_1, x_2, x_3, x_4, x_5 \in X$,

$$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]].$$

The algebraic structures of ternary semigroups were studied by several authors, for example, Sioson studied ideals in ternary semigroups [30], Santiago studied regular ternary semigroups [27], Dixit and Dewan presented quasi-ideals and bi-ideals in ternary semigroups [8], Kar and Maity investigated congruences of ternary semigroups [19] and Iampan studied minimal and maximal lateral ideals of ternary semigroups [12]. Further detail on ternary semigroups, we refer the reader to [8,9,10,11,13,18,22,25,30].

In 1965, Zadeh [31] defined fuzzy sets in order to study mathematical vague situations. Many researchers who are involved in studying, applying, refining and teaching fuzzy sets have successfully applied this theory in many different fields. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [3], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2,4,5] gave the concepts of $(\alpha, \beta)$-fuzzy subgroups by using the “belongs to” relation ($\in$) and ”quasi-coincident with” relation ($q$) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. Keeping this in view, Davvaz in [7] introduced the concept of $(\in, \in \vee q)$-fuzzy sub-nearrings ($R$-subgroups, ideals) of a nearring and investigated some of their interesting properties. Jun and Song [14] discussed general forms of fuzzy interior ideals in semigroups. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [17] and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. Khan et al. [21] characterized ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior ideals and gave some generalized forms of interior ideals in ordered semigroups. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra, for example see [1,15,16,21,28,29].

In this paper, we study the concepts of $(\in, \in \vee q_b)$-fuzzy ideals in ternary semigroups, which is a generalization of the $(\in, \in \vee q)$ fuzzy ideals of a ternary semigroups. We investigate some basic results and properties. We also characterize regular ternary semigroups in terms of these notions.
2 Preliminary Notes

A ternary semigroup $X$ is a non-empty set whose elements are closed under the ternary operation $[\cdot \cdot \cdot ]$ of multiplication and satisfy the associative law defined as $[22]$:

$$[[abc]de] = [a[bed]c] = [ab[cd]e]$$

for all $a, b, c, d, e \in X$.

For simplicity we shall write $[abc]$ as $abc$. For nonempty subsets $A, B$ and $C$ of $X$, let

$$[ABC] := \{[abc] \mid a \in A, b \in B \text{ and } c \in C\}.$$  

An element $a$ of a ternary semigroup $X$ is called regular [11] if there exist elements $x, y \in X$ such that $a = axya$. A ternary semigroup $X$ is regular if every element of $S$ is regular. A non-empty subset $A$ of a ternary semigroup $X$ is called left (resp. right, lateral) ideal of $X$ if $X^2A \subseteq A$ (resp. $AX^2 \subseteq A$, $XAX \subseteq A$)[12]. A non-empty subset $A$ of a ternary semigroup $X$ is called ideal of $X$ if it is left, right and lateral ideal of $X$. A non-empty subset $B$ of a ternary semigroup $X$ is called a ternary subsemigroup [8] if $B^3 \subseteq B$. A subsemigroup $B$ of a ternary subsemigroup $X$ is called a bi-ideal of $X$ if $BXB XB B \subseteq B$ [8]. A subset $Q$ of a ternary semigroup $X$ is called a quasi-ideal [8] of $X$ if $X^2Q \cap X QX \cap QX^2 \subseteq Q$.

Now, we review some fuzzy logic concepts.

A function $f$ from a non-empty set $X$ to the unit interval $[0, 1]$ of real numbers is called a fuzzy subset of $X$, that is $f : X \rightarrow [0, 1]$. For fuzzy subsets $f, g$ of $X$, $f \leq g$ means that for all $a \in X$, $f(a) \leq g(a)$. The symbols $f \land g \land h$ and $f \lor g \lor h$ will mean the following fuzzy subsets of $X$:

$$(f \land g \land h)(a) = f(a) \land g(a) \land h(a),$$

$$(f \lor g \lor h)(a) = f(a) \lor g(a) \lor h(a)$$

for all $a \in X$, where $\land$ denotes min or infimum and $\lor$ denotes max or supremum.

**Definition 2.1** [26] A fuzzy subset $f$ of a ternary semigroup $X$ is a fuzzy ternary subsemigroup of $X$ if $f(abc) \geq f(a) \land f(b) \land f(c)$ for all $a, b, c \in X$.

**Definition 2.2** [26] A fuzzy subset $f$ of a ternary semigroup $X$ is a fuzzy left (resp. lateral, right) ideal of $X$ if $f(abc) \geq f(c)$ (resp. $f(abc) \geq f(b), f(abc) \geq f(a)$) for all $a, b, c \in X$.

**Definition 2.3** [25] A fuzzy subset $f$ of a ternary semigroup $X$ is a fuzzy bi-ideal of $X$ if

(i) $f(abc) \geq f(a) \land f(b) \land f(c)$ and

(ii) $f(abde) \geq f(a) \land f(d) \land f(e)$ for all $a, b, c, d, e \in X$. 

Characterization of ternary semigroups 229
For fuzzy subsets $f, g, h$ of a ternary semigroup $X$, the fuzzy product $f \circ g \circ h$ is defined as

$$(f \circ g \circ h)(a) = \begin{cases} \bigvee_{a=xyz} \{f(x) \land g(y) \land h(z)\} & \text{for } x, y, z \in X, \\ 0 & \text{otherwise} \end{cases}$$

The fuzzy subset $"0"$ and "$X$" of $X$ are defined as follows:

$$0 : X \rightarrow [0,1] | x \mapsto 0(x) = 0,$$

$$X : X \rightarrow [0,1] | x \mapsto X(x) = 1,$$

for all $x \in X$.

**Definition 2.4** [26] A fuzzy subset $f$ of a ternary semigroup $X$ is a fuzzy quasi-ideal of $X$ if

$$f(a) \geq \min \{(f \circ X \circ X)(a), (X \circ f \circ X)(a), (X \circ X \circ f)(a)\}.$$ 

Let $X$ be a ternary semigroup and $f$ a fuzzy subset of $X$, then the set of the form

$$f(y) = \begin{cases} t \neq 0 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is called a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. For a fuzzy point $x_t$ and a fuzzy set $f$ in a set $X$, Pu and Liu [24] introduced the symbol $x_t \alpha f$, where $\alpha \in \{\in, \in \lor \varphi, \in \land \varphi\}$.

For any fuzzy set $f$ in a set $X$, we say that a fuzzy point $x_t$ is

(i) contained in $f$, denoted by $x_t \in f$, [24] if $f(x) \geq t$.

(ii) quasi-coincident with $f$, denoted by $x_t qf$, [24] if $f(x) + t > 1$.

For a fuzzy point $(x, t)$ and a fuzzy set $f$ in $X$, we say that

(iii) $x_t \in \lor qf$ if $x_t \in f$ or $x_t qf$.

(iv) $x_t \alpha f$ if $x_t \alpha f$ does not hold for $\alpha \in \{\in, \in \lor \varphi, \in \land \varphi\}$.

### 3 $(\in, \in \lor \varphi_k)$–fuzzy ideals

In what follows let $X$ denote a ternary semigroup and $k$ an arbitrary element of $(0, 1]$ unless otherwise specified.

Generalizing the concept of $x_t qf$, Jun [14,16] defined $x_t q_k f$, where $k \in [0, 1)$ as $x_t q_k f$ if $f(x) + t + k > 1$. In this section we define the concepts of $(\in, \in \lor \varphi_k)$–fuzzy ideal and $(\in, \in \lor \varphi_k)$–fuzzy bi-ideal of a ternary semigroup $X$ and study some of their basic properties.
Definition 3.1 A fuzzy subset $f$ of $X$ is called $(\in, \in \lor q_k)$-fuzzy ternary subsemigroup of $X$, if for all $a, b, c \in X$ and $t, r, s \in (0, 1]$, 

$$a_t \in f, b_r \in f, c_s \in f \implies (abc)_{t \land r \land s} \in \lor q_k f.$$ 

Example 3.2 (26) Consider the set $Z_5^- = \{0, -1, -2, -3, -4\}$. Then $(Z_5^-, \cdot)$ is a ternary semigroup where ternary multiplication “$\cdot$” is defined as follows:

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Define a fuzzy subset $f$ of $Z_5^-$ as

$$f(0) = 0.9, f(-1) = f(-2) = 0.8, f(-3) = f(-4) = 0.7,$$

then $f$ is an $(\in, \in \lor q_k)$-fuzzy ternary subsemigroup of $Z_5^-$ for any $k \in (0, 1]$.

Definition 3.3 A fuzzy subset $f$ of $X$ is called $(\in, \in \lor q_k)$-fuzzy left (resp. lateral, right) ideal of $X$, if 

$$x_t \in f(\text{resp. } y_r \in f z_s \in f) \implies (xyz)_t \in \lor q_k f \text{ for all } x, y, z \in X \text{ and } t, r, s \in (0, 1].$$

$f$ is called an $(\in, \in \lor q_k)$-fuzzy ideal of $X$, if it is $(\in, \in \lor q_k)$-fuzzy left ideal, $(\in, \in \lor q_k)$-fuzzy lateral ideal and $(\in, \in \lor q_k)$-fuzzy right ideal of $X$.

Definition 3.4 An $(\in, \in \lor q_k)$-fuzzy ternary subsemigroup $f$ of $X$ is called $(\in, \in \lor q_k)$-fuzzy bi-ideal of $X$, if 

$$a_t \in f, c_r \in f, e_s \in f \implies (abcde)_{t \land r \land s} \in \lor q_k f \text{ for all } a, b, c, d, e \in X \text{ and } t, r, s \in (0, 1].$$

Example 3.5 Consider $X = \{0, a, b\}$. Define the multiplication on $X$ as

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Then $(X, \cdot)$ is a ternary semigroup. Define a fuzzy subset $f$ of $X$ as

$$f(0) = 0.9, f(a) = 0.8, f(b) = 0.7,$$

then $f$ is $(\in, \in \lor q_k)$-fuzzy bi-ideal of $X$ for $k = 0.4$.  

Characterization of ternary semigroups
Definition 3.6 A fuzzy subset $f$ of $X$ is called $(\varepsilon, \in \vee q_k)-$fuzzy quasi ideal of $X$, if it satisfies
\[ f(x) \geq \min \left\{ (f \circ X \circ X)(x), (X \circ f \circ X)(x), (X \circ X \circ f)(x), \frac{1 - k}{2} \right\} \]
where $X$ is the fuzzy subset of $X$ mapping every element of $X$ on 1.

Example 3.7 For $Z_5^-$ given in Example 6, define a fuzzy subset $f$ as
\[ f(0) = 0.8, f(-1) = 0.4, f(-2) = 0.5, f(-3) = 0.8, f(-4) = 0.3, \]
then $f$ is $(\varepsilon, \in \vee q_k)-$fuzzy quasi ideal of $Z_5^-$ for $k = 0.6$.

Proposition 3.8 If $B$ be a ternary subsemigroup of a ternary semigroup $X$ then $f$ defined by
\[ f(x) = \begin{cases} \geq \frac{1-k}{2} \text{ if } x \in B \\ 0 \text{ otherwise} \end{cases} \]
is
(i) $(\varepsilon, \in \vee q_k)-$fuzzy ternary subsemigroup of $X$.
(ii) $(\varepsilon, \in \vee q_k)-$fuzzy ternary subsemigroup of $X$.

Proof 3.9 (i) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $x_t, y_r, z_s q \lambda$ then
\[ f(x) + t > 1, f(y) + r > 1 \text{ and } f(z) + s > 1 \]
which shows that $f(x), f(y), f(z) > 0$. Thus $x, y, z \in B$. Since $B$ is ternary subsemigroup so $xyz \in B \implies f(xyz) \geq \frac{1-k}{2}$.

If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then
\[ f(xyz) \geq t \wedge r \wedge \text{sandso}(xyz)_{t \wedge r \wedge s} \in f. \]

If $t \wedge r \wedge s > \frac{1-k}{2}$ then
\[ f(xyz) + t \wedge r \wedge s + k \geq \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \text{andso}(xyz)_{t \wedge r \wedge s} q_k f. \]

Hence $(xyz)_{t \wedge r \wedge s} \in \vee q_k f$.

(ii) Let $x, y, z \in X$ and $t, r, s \in (0, 1]$ be such that $x_t, y_r, z_s \in f$ then
\[ f(x) \geq t > 0, f(y) \geq r > 0 \text{ and } f(z) \geq s > 0. \]
Thus $x, y, z \in B$. Since $B$ is ternary subsemigroup so $xyz \in B \implies f(xyz) \geq \frac{1-k}{2}$.
If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then
\[ f(xyz) \geq t \wedge r \wedge \text{sandso}(xyz)_{t \wedge r \wedge s} \in f. \]

If $t \wedge r \wedge s > \frac{1-k}{2}$ then
\[ f(xyz) + t \wedge r \wedge s + k \geq \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \text{andso}(xyz)_{t \wedge r \wedge s} q_k f. \]

Hence $(xyz)_{t \wedge r \wedge s} \in \vee q_k f$. 

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Theorem 3.10 Let $f$ be a fuzzy subset of $X$ then $f$ is $(\in, \in \lor \bar{q}_k)$–fuzzy ternary subsemigroup of $X$ if and only if

\[ f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\} \] for all $x, y, z \in X$ and $k \in (0, 1]$.

Proof 3.11 $\Rightarrow$. Let $f$ be $(\in, \in \lor \bar{q}_k)$–fuzzy ternary subsemigroup of $X$. Let us suppose on the contrary that there exist $x, y, z \in S$ such that

\[ f(xyz) < \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}. \]

Choose $t \in (0, 1]$ such that

\[ f(xyz) < t \leq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}. \]

then $x_t, y_t, z_t \in f$ but $f(xyz) < t \Rightarrow (xyz)_t \in f$. Also

\[ f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \Rightarrow (xyz)_{t \lor k} = (xyz)_{t \lor k} f. \]

Thus $(xyz)_t \in \lor q_k$, a contradiction. Hence

\[ f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\} \] for all $x, y, z \in X$.

$\Leftarrow$. Conversely assume that $f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}$ for all $x, y, z \in X$. Let $x_t, y_r, z_s \in f$ for some $t, r, s \in (0, 1]$ then $f(x) \geq t, f(y) \geq r$ and $f(z) \geq s$, so

\[ f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\} \geq \min \left\{ t, r, s, \frac{1-k}{2} \right\}. \]

If $t \wedge r \wedge s \leq \frac{1-k}{2}$ then

\[ f(xyz) \geq t \wedge r \wedge s \Rightarrow (xyz)_{t \wedge r \wedge s} \in f. \]

If $t \wedge r \wedge s > \frac{1-k}{2}$ then

\[ f(xyz) + t \wedge r \wedge s + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1 \Rightarrow (xyz)_{t \wedge r \wedge s} \in f. \]

Hence $(xyz)_{t \wedge r \wedge s} \in \lor q_k f$.

Definition 3.12 Let $f$ be a fuzzy subset of $X$, define $U(f; t) = \{ x \in S | f(x) \geq t \}$. We call $U(f; t)$ an upper level cut or upper level set.
Theorem 3.13 A non-empty subset \( f \) of \( X \) is an \((\in, \in \lor \lor q_k)\)-fuzzy ternary subsemigroup of \( X \) if and only if \( U(f; t)(\neq \emptyset) \) is a ternary subsemigroup of \( X \) for all \( t \in (0, \frac{1-k}{2}] \).

Proof 3.14 \( \implies \). Let \( f \) be an \((\in, \in \lor \lor q_k)\)-fuzzy ternary subsemigroup of \( X \). Let \( x, y, z \in U(f; t) \) for some \( t \in (0, \frac{1-k}{2}] \) then \( f(x) \geq t, f(y) \geq t \) and \( f(z) \geq t \). Since \( f \) is an \((\in, \in \lor \lor q_k)\)-fuzzy ternary subsemigroup so

\[
f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\} \geq \min \left\{ t, \frac{1-k}{2} \right\} = t
\]

and so \( xyz \in U(f; t) \). Consequently \( U(f; t) \) is a ternary subsemigroup of \( X \).

\( \Longleftarrow \). Let \( U(f; t) \) be a ternary subsemigroup of \( X \) for all \( t \in (0, \frac{1-k}{2}] \). Suppose that there exist \( x, y, z \in X \) such that

\[
f(xyz) < \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}.
\]

Choosing \( t \in (0, \frac{1-k}{2}] \) such that

\[
f(xyz) < t \leq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}.
\]

Then \( x, y, z \in U(f; t) \) but \( xyz \notin U(f; t) \) which contradicts our supposition. Hence

\[
f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}
\]

and thus \( f \) is \((\in, \in \lor \lor q_k)\)-fuzzy ternary subsemigroup.

Theorem 3.15 Let \( L \) be a left (resp. lateral, right) ideal of \( X \) and \( f \) be a fuzzy subset defined as

\[
f(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}
\]

then

(i) \( f \) is \((q, \in \lor \lor q_k)\)-fuzzy left (resp. lateral, right) ideal of \( X \).

(ii) \( f \) is \((\in, \in \lor \lor q_k)\)-fuzzy left (resp. lateral, right) ideal of \( X \).

Proof 3.16 Proof is similar to Proposition 3.8.
Theorem 3.17  A fuzzy subset $f$ of $X$ is an $(\in, \in \lor q_k)$–fuzzy left (resp. lateral, right) ideal of $X$ if and only if

$$f(xyz) \geq \min \{f(z), \frac{1-k}{2}\}$$

(resp. $f(xyz) \geq \min \{f(y), \frac{1-k}{2}\}$, $f(xyz) \geq \min \{f(x), \frac{1-k}{2}\}$).

for all $x, y, z \in X$ and $k \in (0, 1]$

Proof 3.18  Proof is similar to Theorem 3.10.

By above Theorem, we have the following corollary.

Corollary 3.19  A fuzzy subset $f$ of $X$ is an $(\in, \in \lor q_k)$–fuzzy ideal of $X$ if and only if it satisfies the following

(i)  $f(xyz) \geq \min \{f(z), \frac{1-k}{2}\}$,
(ii)  $f(xyz) \geq \min \{f(y), \frac{1-k}{2}\}$,
(iii)  $f(xyz) \geq \min \{f(x), \frac{1-k}{2}\}$ for all $x, y, z \in X$ and $k \in (0, 1]$.

Theorem 3.20  A non-empty subset $f$ of $X$ is an $(\in, \in \lor q_k)$–fuzzy left (resp. lateral, right) ideal of $X$ if and only if $U(f; t) \neq \emptyset$ is a left (resp. lateral, right) ideal of $X$ for all $t \in (0, \frac{1-k}{2}]$.

Proof 3.21  Proof is similar to Theorem 3.13.

Example 3.22  Let $Z^- = X$ be the set of all negative integers, then $Z^-$ is a ternary semigroup. Let $A = 3X$ then

$$X^2A \subseteq A, AX^2 \subseteq A$$

Hence $A$ is an ideal of $X$. Define $f : X \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} \geq t & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

for any $t \in (0, \frac{1-k}{2})$ then $U(f; t) = \{x \in X | f(x) \geq t\} = \{3X\}$. Since $3X$ is an ideal of $X$ so by Theorem 3.20 $f$ is an $(\in, \in \lor q_k)$–fuzzy ideal of $X$.

Theorem 3.23  Let $B$ be a bi-ideal of $X$ and $f$ be a fuzzy subset defined as

$$f(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

then
(i) $f$ is $(q, \in \forall q_k)$–fuzzy bi-ideal of $X$.

(ii) $f$ is $(\in, \in \forall q_k)$–fuzzy bi-ideal of $X$.

**Proof 3.24** Proof is similar to proposition 3.8.

**Theorem 3.25** A fuzzy subset $f$ of $X$ is $(\in, \in \forall q_k)$–fuzzy bi-ideal of $X$ if and only if it satisfies the following

(i) $f(abc) \geq \min \{f(a), f(b), f(c), \frac{1-k}{2}\}$ for all $a, b, c \in X$ and $k \in (0, 1]$.

(ii) $f(abcd e) \geq \min \{f(a), f(c), f(e), \frac{1-k}{2}\}$ for all $a, b, c, d, e \in X$ and $k \in (0, 1]$.

**Proof 3.26** Proof is similar to Theorem 3.10.

**Theorem 3.27** A non-empty subset $f$ of $X$ is an $(\in, \in \forall q_k)$–fuzzy bi-ideal of $X$ if and only if $U(f; t)(\neq \emptyset)$ is a bi-ideal of $X$ for all $t \in (0, \frac{1-k}{2}]$.

**Proof 3.28** Proof is similar to Theorem 3.13.

**Example 3.29** Let $Z^- = X$ be the set of all negative integers, then $Z^-$ is a ternary semigroup. Let $B = 5X$ then

$$BXBXB = 5XX5XX5X = 125X \subseteq 5X,$$

hence $B$ is a bi-ideal of $X$. Define $f : X \rightarrow [0, 1]$ by

$$f(x) = \begin{cases} \geq ti f x \in B \\ 0 \text{ otherwise} \end{cases}$$

for any $t \in (0, \frac{1-k}{2})$ then $U(f; t) = \{x \in X| f(x) \geq t\} = \{5X\}$. Since $5X$ is a bi-ideal of $X$ so by Theorem 3.27 $f$ is $(\in, \in \forall q_k)$–fuzzy bi-ideal of $X$.

**Theorem 3.30** Let $f$ be $(\in, \in \forall q_k)$–fuzzy quasi ideal of $X$, then the set $f_0 = \{x \in X| f(x) > 0\}$ is a quasi-ideal of $X$.

**Proof 3.31** In order to prove that $f_0$ is a quasi-ideal of $X$, we need to show that $X^2 f_0 \cap f_0 X \cap f_0 X^2 \subseteq f_0$. Let $a \in X^2 f_0 \cap f_0 X \cap f_0 X^2$. This implies that $a \in X^2 f_0$, $a \in X f_0 X$ and $a \in f_0 X^2$. Thus there exist $x_1, y_1, x_2, y_2, x_3, y_3$ in $X$ and $a_1, a_2, a_3$ in $f_0$ such that $a = x_1 y_1 a_1$, $a = x_2 a_2 y_2$ and $a = a_3 x_3 y_3$ thus $f(a_1) > 0$, $f(a_2) > 0$ and $f(a_3) > 0$. Since

$$(f \circ \mathcal{X} \circ \mathcal{X})(a) = \bigvee_{a = pp_1 q_1} \{f(p) \wedge \mathcal{X}(p_1) \wedge \mathcal{X}(q_1)\} \geq f(a_3) \wedge \mathcal{X}(x_3) \wedge \mathcal{X}(y_3) = f(a_3)$$
Similarly \((\mathcal{X} \circ f \circ \mathcal{X})(x) \geq f(a_2)\) and \((\mathcal{X} \circ \mathcal{X} \circ f)(x) \geq f(a_1)\). Thus
\[
f(a) \geq \min \left\{ (\mathcal{X} \circ \mathcal{X} \circ f)(a), (\mathcal{X} \circ f \circ \mathcal{X})(a), (f \circ \mathcal{X} \circ f)(a), \frac{1-k}{2} \right\}
\]
\[
\geq \min \left\{ f(a_1), f(a_2), f(a_3), \frac{1-k}{2} \right\}
\]
\[
> 0 \text{ because } f(a_3) > 0, f(a_2) > 0 \text{ and } f(a_1) > 0
\]
Thus \(a \in f_0\). Hence \(f_0\) is a quasi-ideal of \(X\).

**Lemma 3.32** A non-empty subset \(Q\) of \(X\) is a quasi-ideal of \(X\) if and only if \(C_Q\) is an \((\in, \in \cup q_k)\)-fuzzy quasi-ideal of \(X\).

**Proof 3.33** \(\iff\). Suppose \(Q\) is a quasi-ideal of \(X\) and \(C_Q\) is the characteristic function of \(Q\). If \(x \notin Q\) then \(x \notin X^2Q\) or \(x \notin XQX\) or \(x \notin QX^2\). Thus \((\mathcal{X} \circ \mathcal{X} \circ C_Q)(x) = 0\) or \((\mathcal{X} \circ C_Q \circ \mathcal{X})(x) = 0\) or \((C_Q \circ \mathcal{X} \circ \mathcal{X})(x) = 0\) and so \(\min \left\{ (\mathcal{X} \circ \mathcal{X} \circ C_Q)(x), (\mathcal{X} \circ C_Q \circ \mathcal{X})(x), (C_Q \circ \mathcal{X} \circ \mathcal{X})(x), \frac{1-k}{2} \right\} = 0 = C_Q(x)\). If \(x \in Q\) then \(C_Q(x) = 1 \geq \min \left\{ (\mathcal{X} \circ \mathcal{X} \circ C_Q)(x), (\mathcal{X} \circ C_Q \circ \mathcal{X})(x), (C_Q \circ \mathcal{X} \circ \mathcal{X})(x), \frac{1-k}{2} \right\}\). Hence \(C_Q\) is \((\in, \in \cup q_k)\)-fuzzy.
\(\iff\). Assume that \(C_Q\) is \((\in, \in \cup q_k)\)-fuzzy quasi-ideal of \(X\). Then by Theorem (3.30) \(Q\) is quasi-ideal of \(X\).

**Lemma 3.34** Every \((\in, \in \cup q_k)\)-fuzzy left (resp. lateral, right) ideal of \(X\) is \((\in, \in \cup q_k)\)-fuzzy quasi-ideal of \(X\).

**Proof 3.35** Let \(a \in X\) and let \(f\) be \((\in, \in \cup q_k)\)-fuzzy left ideal then
\[
(\mathcal{X} \circ \mathcal{X} \circ f)(a) = \bigvee_{a=p_1p_2p_3} \{ \mathcal{X}(p_1), \mathcal{X}(p_2), f(p_3) \} = \bigvee_{a=p_1p_2p_3} f(p_3)
\]
This implies that
\[
(\mathcal{X} \circ \mathcal{X} \circ f)(a) \land \frac{1-k}{2} = \left( \bigvee_{a=p_1p_2p_3} f(p_3) \right) \land \frac{1-k}{2}
\]
\[
= \bigvee_{a=p_1p_2p_3} \left( f(p_3) \land \frac{1-k}{2} \right)
\]
\[
\leq \bigvee_{a=p_1p_2p_3} f(p_1p_2p_3) \text{ because } f \text{ is } (\in, \in \cup q_k) \text{ - fuzzy left ideal of } X.
\]
\[
= f(a)
\]
Thus \((\mathcal{X} \circ \mathcal{X} \circ f)(a) \land \frac{1-k}{2} \leq f(a)\). Hence
\[
f(a) \geq (\mathcal{X} \circ \mathcal{X} \circ f)(a) \land \frac{1-k}{2} \geq \min \left\{ (f \circ \mathcal{X} \circ \mathcal{X})(x), (\mathcal{X} \circ f \circ \mathcal{X})(x), (\mathcal{X} \circ \mathcal{X} \circ f)(x), \frac{1-k}{2} \right\}.
\]
Thus \(f\) is \((\in, \in \cup q_k)\)-fuzzy quasi-ideal of \(X\).
Theorem 3.36 Every \((\varepsilon, \in \lor q_k)\)-fuzzy quasi-ideal of \(X\) is \((\varepsilon, \in \lor q_k)\)-fuzzy bi-ideal of \(X\).

Proof 3.37 Suppose that \(f\) is \((\varepsilon, \in \lor q_k)\)-fuzzy quasi-ideal of \(X\). Now

\[
f(xyz) \geq (f \circ \mathcal{X} \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ f \circ \mathcal{X})(xyz) \wedge (\mathcal{X} \circ \mathcal{X} \circ f)(xyz) \wedge \frac{1-k}{2}
\]

\[
= \left[ \bigvee_{xyz=abc} f(a) \wedge \mathcal{X}(b) \wedge \mathcal{X}(c) \right] \wedge \left[ \bigvee_{xyz=pqr} \mathcal{X}(p) \wedge f(q) \wedge \mathcal{X}(r) \right] \wedge \frac{1-k}{2}
\]

\[
\geq [f(x) \wedge \mathcal{X}(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge f(y) \wedge \mathcal{X}(z)] \wedge [\mathcal{X}(x) \wedge \mathcal{X}(y) \wedge f(z)] \wedge \frac{1-k}{2}
\]

\[
= f(x) \wedge f(y) \wedge f(z) \wedge \frac{1-k}{2}
\]

so \(f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\} \). In the same way it can be proved that \(f(abcde) \geq \min \left\{ f(a), f(c), f(e), \frac{1-k}{2} \right\} \). Thus \(f\) is \((\varepsilon, \in \lor q_k)\)-fuzzy bi-ideal of \(X\).

Theorem 3.38 Every \((\varepsilon, \in \lor q_k)\)-fuzzy left (resp. lateral, right) ideal of \(X\) is \((\varepsilon, \in \lor q_k)\)-fuzzy bi-ideal of \(X\).

Proof 3.39 Let \(f\) be \((\varepsilon, \in \lor q_k)\)-fuzzy left (resp. lateral, right) ideal of \(X\) then

\[
f(abc) \geq \min \left\{ f(c), \frac{1-k}{2} \right\} \text{ for all } a, b, c \in X.
\]

Suppose that there exists \(x, y, z \in X\) such that

\[
f(xyz) < \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}.
\]

Choose \(t \in (0, 1]\) such that

\[
f(xyz) < t \leq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}
\]

then \(z_t \in f\) but \((xyz)_t \notin f\). Also

\[
f(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1,
\]

which implies \((xyz)_t \lor q_k f\). Thus \((xyz)_t \in \lor q_k f\), which contradicts the hypothesis. Thus

\[
f(xyz) \geq \min \left\{ f(x), f(y), f(z), \frac{1-k}{2} \right\}.
\]
Similarly let there exist \(a, b, c, d, e \in X\) such that

\[
f(abcde) < \min \left\{ f(a), f(c), f(e), \frac{1-k}{2} \right\}.
\]

Choose \(t \in (0, 1]\) such that

\[
f(a(bcd)e) < t \leq \min \left\{ f(a), f(c), f(e), \frac{1-k}{2} \right\}
\]

then \(e_t \in f\) but \((a(bcd)e)_t \not\in f\). Also

\[
f(abcde) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1
\]

for all \(a, b, c, d, e \in X\), thus \((abcde)_{\frac{1-k}{2}} f\). Hence \((abcde)_{\frac{1-k}{2}} f\). A contradiction, hence

\[
f(abcde) \geq \min \left\{ f(a), f(c), f(e), \frac{1-k}{2} \right\}.
\]

Consequently \(f\) is \((\varepsilon, \varepsilon \in \vee_{q_k}) -\)fuzzy bi-ideal of \(X\).

## 4 Regular Ternary Semigroups

In this section we characterize regular ternary semigroups by the properties of their \((\varepsilon, \varepsilon \in \vee_{q_k})\) -fuzzy ideals and \((\varepsilon, \varepsilon \in \vee_{q_k})\) -fuzzy bi-ideals.

### Definition 4.1
Let \(f, f_1\) and \(f_2\) be fuzzy subsets of \(X\). We define \(f_k, f \land_k f_1 \land_k f_2, f \lor_k f_1 \lor_k f_2, f \circ_k f_1 \circ_k f_2\) as follows.

\[
f_k(x) = f(x) \land \frac{1-k}{2}
\]

\[
(f \land_k f_1 \land_k f_2)(x) = (f \land f_1 \land f_2)(x) \land \frac{1-k}{2}
\]

\[
(f \lor_k f_1 \lor_k f_2)(x) = (f \lor f_1 \lor f_2)(x) \land \frac{1-k}{2}
\]

\[
(f \circ_k f_1 \circ_k f_2)(x) = (f \circ f_1 \circ f_2)(x) \land \frac{1-k}{2}
\]

for all \(x \in X\).

### Lemma 4.2 [28]
Let \(f, f_1\) be fuzzy subsets of a semigroup \(S\). Then the following hold.

1. \((f \land_k f_1) = (f_k \land f_{1k})\)
2. \((f \lor_k f_1) = (f_k \lor f_{1k})\)
3. \((f \circ_k f_1) = (f_k \circ f_{1k})\).

### Lemma 4.3
Let \(f, f_1\) and \(f_2\) be fuzzy subsets of \(X\). Then the following hold.
\( (i) \ (f \wedge_k f_1 \wedge_k f_2) = (f_k \wedge f_{1k} \wedge f_{2k}) \)

\( (ii) \ (f \vee_k f_1 \vee_k f_2) = (f_k \vee f_{1k} \vee f_{2k}) \)

\( (iii) \ (f \circ_k f_1 \circ_k f_2) = (f_k \circ f_{1k} \circ f_{2k}) \).

**Proof 4.4** Proof is a consequence of Lemma 4.2.

**Lemma 4.5** [28] Let \( A, B \) be non-empty subsets of a semigroup \( S \). Then the following hold.

1. \((C_A \wedge_k C_B) = (C_{A \wedge B})_k\)
2. \((C_A \vee_k C_B) = (C_{A \vee B})_k\)
3. \((C_A \circ_k C_B) = (C_{AB})_k\).

**Lemma 4.6** Let \( A, B, D \) be non-empty subsets of a ternary semigroup \( X \). Then the following hold.

1. \((C_A \wedge_k C_B \wedge_k C_D) = (C_{A \wedge B \wedge D})_k\)
2. \((C_A \vee_k C_B \vee_k C_D) = (C_{A \vee B \vee D})_k\)
3. \((C_A \circ_k C_B \circ_k C_D) = (C_{ABD})_k\).

**Proof 4.7** Follows from Lemma 4.5.

**Lemma 4.8** Let \( L \) be a non-empty subset of a ternary semigroup \( X \) then \( L \) is left (resp. lateral, right) ideal of \( X \) if and only if \((C_L)_k\) is \((\in, \in \vee q)\)-fuzzy left (resp. lateral, right) ideal of \( X \).

**Proof 4.9** \( \implies \). Let \( L \) be a left ideal of a ternary semigroup \( X \) then by Theorem 3.15 \((C_L)_k\) is \((\in, \in \vee q)\)-fuzzy left ideal of \( X \).

\( \iff \). Suppose \((C_L)_k\) be \((\in, \in \vee q)\)-fuzzy left ideal of \( X \). Let \( z \in L \) then \((C_L)_k(z) = \frac{1-k}{2} \). So \( z = \frac{1-k}{2} \in (C_L)_k \). Since \((C_L)_k\) is \((\in, \in \vee q)\)-fuzzy left ideal of \( X \) so, we have,

\[
(C_L)_k(xyz) \geq \frac{1-k}{2} \quad \text{or} \quad (C_L)_k(xyz) + \frac{1-k}{2} + k > 1 \quad \text{for all} \ x, y \in X.
\]

If \((C_L)_k(xyz) + \frac{1-k}{2} + k > 1 \) then \((C_L)_k(xyz) > \frac{1-k}{2} \), thus \((C_L)_k(xyz) \geq \frac{1-k}{2} \)

which gives \((C_L)_k(xyz) = \frac{1-k}{2} \), and hence \( xyz \in L \) for all \( x, y \in X \) and \( z \in L \). Consequently \( L \) is a left ideal of \( X \).

**Lemma 4.10** Let \( B \) be a non-empty subset of a ternary semigroup \( X \) then \( B \) is bi-ideal of \( X \) if and only if \((C_B)_k\) is \((\in, \in \vee q)\)-fuzzy bi-ideal of \( X \).
Characterization of ternary semigroups

**Proof 4.11** Similar to Theorem 4.8.

**Proposition 4.12** Let \( f \) be an \((\in, \in, \lor q_k)\)–fuzzy left (resp. lateral, right) ideal of \( X \) then \( f_k \) is a fuzzy left (resp. lateral, right) ideal of \( X \).

**Proof 4.13** Let \( f \) be an \((\in, \in, \lor q_k)\)–fuzzy left ideal of \( X \) then for all \( a, b, c \in X \) we have

\[
f(abc) \geq \min\left\{f(c), \frac{1-k}{2}\right\}.
\]

Which implies that

\[
f(abc) \land \frac{1-k}{2} \geq \min\left\{f(c), \frac{1-k}{2}\right\} \Rightarrow f_k(abc) \geq f_k(c),
\]

so \( f_k \) is fuzzy left ideal of \( X \).

**Theorem 4.14** \([8]\) A ternary semigroup \( X \) is regular if and only if for every left ideal \( L \), lateral ideal \( T \) and right ideal \( R \) of \( X \) we have \( R \cap T \cap L = RT L \).

**Theorem 4.15** For a ternary semigroup \( X \) the following are equivalent.

(i) \( X \) is regular.

(ii) \((f \land_k f_1 \land_k f_2) = (f \circ_k f_1 \circ_k f_2)\) for every \((\in, \in, \lor q_k)\)–fuzzy right ideal \( f \), \((\in, \in, \lor q_k)\)–fuzzy lateral ideal \( f_1 \) and \((\in, \in, \lor q_k)\)–fuzzy left ideal \( f_2 \) of \( X \).

**Proof 4.16** (i) \(\Rightarrow\) (ii) Let \( X \) be a regular ternary semigroup and let \( f \) be \((\in, \in, \lor q_k)\)–fuzzy right ideal, \( f_1 \) be \((\in, \in, \lor q_k)\)–fuzzy lateral ideal and \( f_2 \) be \((\in, \in, \lor q_k)\)–fuzzy left ideal of \( X \). Now

\[
(f \circ_k f_1 \circ_k f_2)(a) = (f \circ f_1 \circ f_2)(a) \land \frac{1-k}{2}
\]

\[
= \left( \bigvee_{a=bcd} f(b) \land f_1(c) \land f_2(d) \right) \land \frac{1-k}{2}
\]

\[
= \bigvee_{a=bcd} \left\{ f(b) \land f_1(c) \land f_2(d) \land \frac{1-k}{2} \right\}
\]

\[
= \bigvee_{a=bcd} \left\{ \left( f(b) \land \frac{1-k}{2} \right) \land \left( f_1(c) \land \frac{1-k}{2} \right) \land \left( f_2(d) \land \frac{1-k}{2} \right) \right\}
\]

\[
\leq \bigvee_{a=bcd} \left\{ f(bcd) \land f_1(bcd) \land f_2(bcd) \land \frac{1-k}{2} \right\}
\]

\[
= f(a) \land f_1(a) \land f_2(a) \land \frac{1-k}{2}
\]

\[
= (f_k \land f_{1k} \land f_{2k})(a),
\]
so \((f \circ_k f_1 \circ_k f_2) \leq (f \wedge_k f_1 \wedge_k f_2)\).

On the other hand since \(X\) is regular so for \(a \in X\) there exist \(x, y \in X\) such that \(a = axaya = axayaxaya\). Thus

\[
(f \circ_k f_1 \circ_k f_2)(a) = (f \circ f_1 \circ f_2)(a) \wedge \frac{1-k}{2}
\]

\[
= \left( \bigvee_{a=bcd} f(b) \wedge f_1(c) \wedge f_2(d) \right) \wedge \frac{1-k}{2}
\]

\[
= \bigvee_{a=bcd} \left\{ f(b) \wedge f_1(c) \wedge f_2(d) \wedge \frac{1-k}{2} \right\}
\]

\[
\geq f(axa) \wedge f_1(yax) \wedge f_2(aya) \wedge \frac{1-k}{2}
\]

\[
\geq \left( f(a) \wedge \frac{1-k}{2} \right) \wedge \left( f_1(a) \wedge \frac{1-k}{2} \right) \wedge \left( f_2(a) \wedge \frac{1-k}{2} \right) \wedge \frac{1-k}{2}
\]

\[
= f(a) \wedge \frac{1-k}{2} \wedge f_1(a) \wedge \frac{1-k}{2} \wedge f_2(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2}
\]

\[
= f(a) \wedge f_1(a) \wedge f_2(a) \wedge \frac{1-k}{2}
\]

\[
= (f_k \wedge f_1k \wedge f_2k)(a).
\]

Hence \((f \circ_k f_1 \circ_k f_2) = (f \wedge_k f_1 \wedge_k f_2)(a)\).

(ii) \(\implies\) (i). Suppose that \(R, T\) and \(L\) be right, lateral and left ideals of \(X\) respectively then by Lemma 4.8 \((C_R)_k, (C_T)_k\) and \((C_L)_k\) are \((\in, \in \vee q_k)\)–fuzzy right ideal, \((\in, \in \vee q_k)\)–fuzzy lateral ideal and \((\in, \in \vee q_k)\)–fuzzy left ideal of \(X\) respectively. Thus we have

\[
(C_{RTL})_k = C_R \circ_k C_T \circ_k C_L \text{ by Lemma 4.6}
\]

\[
= C_R \wedge_k C_T \wedge_k C_L \text{ by (ii) above}
\]

\[
= (C_{RTL\wedge L})_k \text{ by Lemma 4.6}
\]

Thus \(RTL = R \cap T \cap L\), and thus \(X\) is regular.

**Theorem 4.17** The following assertions for a ternary semigroup \(X\) are equivalent.

(i) \(X\) is regular.

(ii) \(f_k = f \circ_k X \circ_k f \circ_k X \circ_k f\) for every \((\in, \in \vee q_k)\)–fuzzy bi-ideal \(f\) of \(X\).

**Proof 4.18** (i) \(\implies\) (ii) Let \(X\) be regular ternary semigroup and \(f\) be \((\in, \in \vee q_k)\)–fuzzy bi-ideal of \(X\). Since \(X\) is regular so for \(a \in X\) there exist \(x, y \in X\)
such that \( a = axaya. \) Now

\[
(f \circ_k X \circ_k f \circ_k X \circ_k f)(a) = \left( \bigvee_{a=bcd} (f \circ_k X \circ_k f)(b) \land X(c) \land f(d) \right) \land \frac{1-k}{2}
\]

\[= \bigvee_{a=bcd} \left( (f \circ_k X \circ_k f)(b) \land X(c) \land f(d) \land \frac{1-k}{2} \right)
\]

\[= \bigvee_{a=bcd} \left\{ \left( \bigvee_{b=xyz} f(x) \land X(y) \land f(z) \land \frac{1-k}{2} \right) \land X(c) \land f(d) \land \frac{1-k}{2} \right\}
\]

\[= \bigvee_{a=(xyz)cd} \left( f(x) \land X(y) \land f(z) \land X(c) \land f(d) \land \frac{1-k}{2} \right)
\]

\[\geq f(a) \land X(x) \land f(a) \land X(y) \land f(a) \land \frac{1-k}{2}
\]

\[= f(a) \land 1 \land f(a) \land 1 \land f(a) \land \frac{1-k}{2}
\]

\[= f(a) \land \frac{1-k}{2}
\]

\[= f_k(a)
\]

So \( (f \circ_k X \circ_k f \circ_k X \circ_k f) \geq f_k. \) On the other hand

\[
(f \circ_k X \circ_k f \circ_k X \circ_k f)(a) = \left( \bigvee_{a=abcd} (f \circ_k X \circ_k f)(b) \land X(c) \land f(d) \right) \land \frac{1-k}{2}
\]

\[= \bigvee_{a=abcd} \left( (f \circ_k X \circ_k f)(b) \land X(c) \land f(d) \land \frac{1-k}{2} \right)
\]

\[= \bigvee_{a=abcd} \left\{ \left( \bigvee_{b=xyz} f(x) \land X(y) \land f(z) \land \frac{1-k}{2} \right) \land X(c) \land f(d) \land \frac{1-k}{2} \right\}
\]

\[= \bigvee_{a=(xyz)cd} \left( f(x) \land X(y) \land f(z) \land X(c) \land f(d) \land \frac{1-k}{2} \right)
\]

\[= \bigvee_{a=(xyz)cd} \left( f(x) \land f(z) \land f(d) \land \frac{1-k}{2} \right)
\]

\[\leq \bigvee_{a=(xyz)cd} f(xyzcd) \land \frac{1-k}{2} \text{ as } f \text{ is } (\in, \in \lor q_k) - \text{fuzzy bi-ideal of } X
\]

\[= f(a) \land \frac{1-k}{2}
\]

\[= f_k(a)
\]
Hence we get \((f \circ_k C X \circ_k f \circ_k X \circ_k f) \leq f_k\). Thus
\[
(f \circ_k X \circ_k f \circ_k X \circ_k f) = f_k.
\]

\((ii) \implies (i)\) Let \(B\) be a bi-ideal of \(X\) then by Lemma 4.10 \((C_B)_k\) is \((\varepsilon, \in \lor \eta_k)-\text{fuzzy bi-ideal of } X\). Let \(a \in B\) then \((C_B)_k(a) = \frac{1-k}{2}\). Now by \((ii)\) above
\[
(C_B)_k(a) = \left( C_B \circ_k X \circ_k C_B \circ_k X \circ_k C_B \right)(a)
\]
\[
= \bigvee_{a=bfg} \left( C_B \circ_k X \circ_k C_B \right)(b) \land X(f) \land C_B(g) \land \frac{1-k}{2}
\]
\[
= \bigvee_{a=bfg} \left\{ \bigvee_{b=cde} (C_B(c) \land X(d) \land C_B(e) \land \frac{1-k}{2}) \land X(f) \land C_B(g) \land \frac{1-k}{2} \right\}
\]
\[
= \bigvee_{a=(cde)fg} C_B(c) \land X(d) \land C_B(e) \land X(f) \land C_B(g) \land \frac{1-k}{2}
\]
\[
= \bigvee_{a=(cde)fg} \left( C_B(c) \land \frac{1-k}{2} \right) \land \left( C_B(e) \land \frac{1-k}{2} \right) \land \left( C_B(g) \land \frac{1-k}{2} \right)
\]
\[
= \bigvee_{a=(cde)fg} \left( C_B(c) \right)_k \land \left( C_B(e) \right)_k \land \left( C_B(g) \right)_k
\]
\[
= \frac{1-k}{2} \text{ as } (C_B)_k(a) = \frac{1-k}{2}
\]

Which is only possible if \(c, e, g \in B\). Thus \(cdefg \in BXBXB\) and hence \(a \in BXBXB\) so \(B \subseteq BXBXB\) but \(BXBXB \subseteq B\) so \(X\) is regular.

References


Characterization of ternary semigroups


Anwar Zeb, Gul Zaman, Inayat Ali Shah, Asghar Khan


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