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Boundedness for Hardy-Littlewood Maximal Operator and Hilbert Transform in Weighted Grand L^∞ Space

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Abstract

In this paper we obtain boundedness results for Hardy-Littlewood maximal operator and Hilbert transform in weighted grand L^∞ space $L_w^{\infty)}(\Omega)$ with the weight $w \in A_\infty$.

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1 Introduction

Let $I = (0, 1)$. The classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{I \supset J \ni x} \frac{1}{|J|} \int_J |f(t)| dt, \quad x \in (0, 1),$$

where the supremum extends over all non-degenerate intervals, contained in I, containing x and $|J|$ denoted the Lebesgue measure of J.

The Hilbert transform H is defined by

$$Hf(x) = \text{p.v.} \int_I \frac{f(t)}{x-t} dt, \quad x \in (0,1),$$

where p.v. stands for principal value.

For some properties of Hardy-Littlewood maximal operator and Hilbert transform, see [1].

Let w be a weight on I, that is, a positive almost everywhere integrable function on I. Let $1 < p < \infty$. We say that a weight w belongs to the Muckenhoupt class $A_p(I)$ ($w \in A_p(I)$) if

$$A_p(w, I) = \sup_J \left(\frac{1}{|J|} \int_J w(x) dx \right) \left(\frac{1}{|J|} \int_J w^{1-p'(t)} dt \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $J \subset I$, and $p' : \frac{1}{p'} + \frac{1}{p} = 1$. We define $A_\infty = \bigcup\limits_{1<p<\infty} A_p$. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$. The weighted Lebesgue spaces with respect to the measure $w(x)dx$ are denoted by $L^p_w(I)$ with $1 \le p < \infty$.

Let w be a weight. The weighted grand L^∞ space $L^{\infty)}_w(I)$ is defined in [2] by

$$L^{\infty)}_w(I) = \left\{ f(x) \in \bigcap_{1<p<\infty} L^p_w(I) : \|f\|_{L^p_w(I)} < \infty \right\},$$

where

$$\|f\|_{L^p_w(I)} = \sup_{1<p<\infty} \frac{1}{p} \left(\frac{1}{w(I)} \int_I |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

For some properties of the weighted grand L^∞_w spaces, we refer the reader to [3].

The aim of this paper is to derive boundedness for the Hardy-Littlewood maximal operator M and the Hilbert transform H in the weighted grand L^∞ space $L^{\infty)}_w(I)$.

2 Main Results

In order to prove the main theorems of this paper, we need a preliminary lemma, which can be found in [3].

Lemma 2.1. *If $w \in A_\infty$, then there exists $q \in (1, \infty)$ such that $w \in A_q$.*

We first consider boundedness of the Hardy-Littlewood maximal operator in weighted grand L^∞ space $L^{\infty)}_w(I)$. In the framework of the standard Lebesgue spaces, it is well-known that

$$\|Mf\|_{p,w} \le c\|f\|_{p,w} \tag{2.1}$$

is true if and only if $w \in A_p$, $1 < p < \infty$.

Theorem 2.1. *Let $w \in A_\infty$. Then*

$$\|Mf\|_{\infty),w} \leq cq\|f\|_{\infty),w}.$$

Proof. By Lemma 2.1, Hölder inequality and (2.1), we have

$$\begin{aligned}
& \|Mf\|_{\infty),w} \\
= & \sup_{1<p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}} \\
= & \max\left\{\sup_{1<p<q} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}, \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}\right\} \\
\leq & \max\left\{\sup_{1<p<q} \frac{1}{p}\left(\frac{1}{w(I)}\int_I \left(|Mf(x)|^p w(x)^{\frac{p}{q}}\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \left(\frac{1}{w(I)}\int_I \left(w(x)^{\frac{q-p}{q}}\right)^{\frac{q}{q-p}} dx\right)^{\frac{q-p}{qp}}, \right. \\
& \left. \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}\right\} \\
= & \max\left\{\sup_{1<p<q} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^q w(x)dx\right)^{\frac{1}{q}}, \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}\right\} \\
\leq & \max\left\{\sup_{1<p<q} \frac{q}{p} \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}, \right. \\
& \left. \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}}\right\} \\
= & q \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |Mf(x)|^p w(x)dx\right)^{\frac{1}{p}} \\
\leq & cq \sup_{q\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |f(x)|^p w(x)dx\right)^{\frac{1}{p}} \\
\leq & cq \sup_{1\leq p<\infty} \frac{1}{p}\left(\frac{1}{w(I)}\int_I |f(x)|^p w(x)dx\right)^{\frac{1}{p}} \\
= & cq\|f\|_{\infty),w}.
\end{aligned}$$

This ends the proof of Theorem 2.1.

We next consider boundedness of Hilbert transform in weighted grand L^∞ space L^∞ space $L_w^{\infty)}(I)$. It is known that a necessary and sufficient condition for the boundedness of the Hilbert transform in L_w^p is that w satisfies the Muckenhoupt condition A_p. That is,

$$\|Hf\|_{p,w} \leq c\|f\|_{p,w} \tag{2.2}$$

holds true if and only if $w \in A_p$.

Theorem 2.2. *Let $w \in A_\infty$. Then*

$$\|Hf\|_{\infty),w} \leq cq\|f\|_{\infty),w}.$$

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1 with H in place of M. We omit the details.

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