Blow-up and global existence for the periodic two-component $\mu$-Hunter-Saxton system

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Abstract
The two-component $\mu$-Hunter-Saxton system is considered in the spatially periodic setting. Firstly, a wave-breaking criterion is derived by employing the localization analysis of the transport equation theory. Using this criterion, then we prove the global existence of strong solutions for the system.

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1 Introduction

In this article, we will consider the periodic two-component $\mu$-Hunter-Saxton system derived by Zuo [11]

\[
\begin{align*}
\mu(u_t) - u_{txx} &= -2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx} + \rho\rho_x - \gamma_1u_{xxx}, \quad t > 0, x \in \mathbb{R}, \\
\rho_t &= (u\rho)_x + 2\gamma_2\rho_x, \quad t > 0, x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\
\rho(0, x) &= \rho_0(x), \quad x \in \mathbb{R}, \\
u(t, x + 1) &= u(t, x), \quad t > 0, x \in \mathbb{R}, \\
\rho(t, x + 1) &= \rho(t, x), \quad t > 0, x \in \mathbb{R},
\end{align*}
\]

where $u(t, x)$ and $\rho(t, x)$ are time-dependent functions on the unit circle $S = \mathbb{R}/\mathbb{Z}$, $\mu(u) = \int_S u\,dx$ denotes its mean and $\gamma_i \in \mathbb{R}, i = 1, 2$. It is shown in [11] that system (1) is an Euler equation with bi-Hamilton structure

\[
\Gamma_1 = \begin{pmatrix}
\partial_x & 0 \\
0 & \partial_x
\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix}
A(u)\partial_x + \partial_x A(u) - \gamma_1\partial_x^3 & \rho\partial_x \\
\partial_x \rho & 2\gamma_2\partial_x
\end{pmatrix},
\]
where $A(u) = \mu(u) - u_{xx}$, and also be viewed as a bi-variational equation. Moreover, for $\gamma_i = 0$, $i = 1, 2$, system (1) has a Lax pair given by
\[
\psi_{xx} = \lambda (A(u) - \lambda^2 \rho^2) \psi, \quad \psi_t = (u - \frac{1}{2\lambda}) \psi_x - \frac{1}{2} u_x \psi,
\]
where $\lambda$ is a spectral parameter (see [11]).

In fact, system (1) is a generalization of the generalized Hunter-Saxton equation [5, 6]
\[
\mu(u_t) - u_{txx} = -2 \mu(u) u_x + 2 u_x u_{xx} + uu_{xxx},
\]
which describes the geodesic flow on $D^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product $\langle u, v \rangle = \mu(u) \mu(v) + \int_\mathbb{S} u_x v_x dx$, and models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic nematic field and self-interaction. Here, the solution $u(t, x)$ denotes the director field of a nematic liquid crystal. It was observed in [5, 6] that the $\mu$-Hunter-Saxton equation is formally integrable, has bi-Hamiltonian structure and infinite hierarchy of conservation laws. Further, the development of singularities in finite time and geometric descriptions of the system from nonstretching invariant curve flows in centro-equiaffine geometries, pseudo-spherical surfaces and affine surfaces are described by Fu et al [3].

Recently, Liu and Yin [7, 8] investigated the Cauchy problem for system (1). In [7], the local well-posedness and several precise blow-up criteria for the system were obtained. Under the conditions $\mu_0 = 0$ and $\mu_0 \neq 0$, the sufficient conditions of blow-up solutions were presented. The global existence for strong solution for system (1) in the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s = 2$ is also given [7], and in [8], existence of global weak solution is established for the periodic two-component $\mu$-Hunter-Saxton system. The objective of the present paper is to focus mainly on wave-breaking criterion and several sufficient conditions of blow-up solutions.

Motivated by the works in [4, 9], in the present paper, the localization analysis in the transport equation theory is employed to derive a new wave-breaking criterion of solutions for the system (1) in the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$. It implies that the wave-breaking criterion is determined only by the slope of the component $u$ of solution definitely. Further, by using the wave-breaking criterion, we also present a sufficient condition for the existence of global strong solutions. Motivated by the work in [2]. These results obtained in this paper are new and different from those in Liu and Yin’s work [7].

The rest of this paper is organized as follows. Section 2 states several properties for the periodic two-component $\mu$-Hunter-Saxton system and gives several lemmas. In Section 3, we present a wave-breaking criterion in the Sobolev space $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$. An improved result of the global existence of solutions for system (1) is given in Section 4.
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2 Preliminary

Lemma 2.1 ([7]) Given \( z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), \ s \geq 2 \), then there exists a maximal \( T = T(\| z_0 \|_{H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})}) > 0 \) and an unique solution \( z = (u, \rho) \) to system (1) such that

\[
z = z(\cdot, z_0) \in C([0, T); H^s(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S})).
\]

Lemma 2.2 ([1]) For every \( f(x) \in H^1(a, b) \) periodic and with zero average, i.e. such that

\[
\int_a^b f(x) \, dx = 0,
\]

it holds that

\[
\int_a^b f^2(x) \, dx \leq \left( \frac{b-a}{2\pi} \right)^2 \int_a^b |f'(x)|^2 \, dx,
\]

and equality holds if and only if

\[
f(x) = A \cos\left(\frac{2\pi x}{b-a}\right) + B \sin\left(\frac{2\pi x}{b-a}\right).
\]

Integrating the first equation of system (1) over the circle \( \mathbb{S} = \mathbb{R}/\mathbb{Z} \) and noting the periodicity of \( u \), we have \( \mu(u_t) = 0 \). Making use of system (1), we have that \( \int_\mathbb{S} (u_x^2 + \rho^2) \, dx \) is conserved in time (see [7]). In what follows we denote

\[
\mu_0 = \mu(u_0) = \mu(u) = \int_\mathbb{S} u(t, x) \, dx
\]

and

\[
\mu_1 = \left( \int_\mathbb{S} u_x^2(t, x) + \rho^2(t, x) \, dx \right)^{\frac{1}{2}} = \left( \int_\mathbb{S} u_x^2(0, x) + \rho^2(0, x) \, dx \right)^{\frac{1}{2}}.
\]

Then \( \mu_0 \) and \( \mu_1 \) are constants and independent of time \( t \).

Notice that \( \int_\mathbb{S} (u(t, x) - \mu_0) \, dx = \mu_0 - \mu_0 = 0 \). From Lemma 2.5, we get

\[
\max_{x \in \mathbb{S}} |u(t, x) - \mu_0|^2 \leq \frac{1}{12} \int_\mathbb{S} u_x^2(t, x) \, dx \leq \frac{1}{12} \int_\mathbb{S} u_x^2(t, x) + \rho^2(t, x) \, dx
\]

\[
= \frac{1}{12} \int_\mathbb{S} u_x^2(0, x) + \rho^2(0, x) \, dx = \frac{1}{12} \mu_1^2,
\]

which implies that the amplitude of wave remains bounded in any time. Namely, we have

\[
\| u(t, \cdot) \|_{L^\infty(\mathbb{S})} - |\mu_0| \leq \| u(t, \cdot) - \mu_0 \|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \mu_1,
\]
which results in
\[ \| u(t, \cdot) \|_{L^\infty(S)} \leq |\mu_0| + \frac{\sqrt{3}}{6} \mu_1. \]  

In fact, the initial-value problem (1) can be recast in the following
\[
\begin{aligned}
&u_t - (u + \gamma_1)u_x = A^{-1} \partial_x (2\mu_0 u + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2), \quad t > 0, x \in \mathbb{R}, \\
&\rho_t - (u + 2\gamma_2)\rho_x = \rho u_x, \quad t > 0, x \in \mathbb{R}, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
&\rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \\
&u(t, x + 1) = u(t, x), \quad t > 0, x \in \mathbb{R}, \\
&\rho(t, x + 1) = \rho(t, x), \quad t > 0, x \in \mathbb{R},
\end{aligned}
\]  

where \( A = \mu - \partial_x^2 \) is an isomorphism between \( H^s \) and \( H^{s-2} \) with the inverse \( \nu = A^{-1} \omega \) given explicitly by
\[
\nu(x) = \left( \frac{x^2}{2} - x + \frac{13}{12} \right) \mu(\omega) + \left( x - \frac{1}{2} \right) \int_0^1 \int_0^y \omega(s) ds dy \\
- \int_0^x \int_0^y \omega(s) ds dy + \int_0^1 \int_0^y \int_0^r \omega(r) dr dy.
\]  

Commuting \( A^{-1} \) and \( \partial_x \), we get
\[
A^{-1} \partial_x \omega(x) = (x - \frac{1}{2}) \int_0^1 \omega(x) - \int_0^x \omega(y) dy + \int_0^1 \int_0^x \omega(y) dy dx
\]  

and
\[
A^{-1} \partial_x^2 \omega(x) = -\omega(x) + \int_0^1 \omega(x) dx.
\]  

Note that if \( f \in L^2(S) \), we have \( A^{-1} f = (\mu - \partial_x^2)^{-1} f = g * f \), where we denotes by \( * \) convolution and \( g \) is the Green’s function of the operator \( A^{-1} \), given by
\[
g(x) = \frac{1}{2} (x - \frac{1}{2})^2 + \frac{23}{24},
\]  

and the derivative of \( g \) can be assigned
\[
g_x(x) = \begin{cases} 
0, & x = 0, \\
\frac{x - 1}{2}, & x \in (0, 1).
\end{cases}
\]  

Now, consider the initial value problem for the Lagrangian flow map:
\[
\begin{aligned}
&\eta_t = u(t, -\eta) + 2\gamma_2, \quad t \in [0, T), \\
&\eta(0, x) = x, \quad x \in \mathbb{R},
\end{aligned}
\]
where $u$ denotes the first component of the solution $z = (u, \rho)$ to system (1).

Applying classical results from ordinary differential equations, one can obtain the result.

**Lemma 2.3** ([7]) Let $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$, $s \geq 2$. Then Eq. (14) has an unique solution $\eta \in C^1([0, T) \times \mathbb{R}; \mathbb{R})$. Moreover, the map $\eta(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$
\eta_x(t, x) = \exp(- \int_0^t u_x(s, -\eta(s, x)) ds) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}. \quad (15)
$$

**Lemma 2.4** ([7]) Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$ and let $T > 0$ be the maximal existence time of the corresponding solution $z = (u, \rho)$ to system (1). Then it has

$$
\rho(t, -\eta(t, x))\eta_x(t, x) = \rho_0(-x), \quad (t, x) \in [0, T) \times \mathbb{R}. \quad (16)
$$

### 3 Wave-breaking criterion

**Theorem 3.1** Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$, and $z = (u, \rho)$ be the corresponding solution to (1). Assume that $T > 0$ is the maximal existence time. Then

$$
T < \infty \Rightarrow \int_0^T \| u_x \|_{L^\infty(\mathbb{S})} d\tau = \infty. \quad (17)
$$

**Proof.** The proof is similar with that of Theorem 1 in [12], hence, we omit the proof of Theorem 3.1.

### 4 Global existence

In this section, using the above criterion of wave breaking, we provide a sufficient condition for the global solution of system (1).

**Theorem 4.1** Let $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ with $s \geq 2$ and let $T$ be the maximal time of existence. If $\gamma_1 = 2\gamma_2$ and $\rho_0(-x) \neq 0$, then the solution $z = (u, \rho)$ of system (1) with initial value $z_0 = (u_0, \rho_0)$ is global.

**Proof.** Applying a simple density argument, we only need to consider the case $s = 3$. From Lemma 2.3, we know that $\eta(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$. Setting $M(t) = u_x(t, -\eta(t, x))$ and $\psi(t) = \rho(t, -\eta(t, x))$
and applying the assumption of theorem and (14), system (8) becomes the following ordinary differential equations

\[ M'(t) = \frac{1}{2} M^2 - \frac{1}{2} \psi^2(t) + f(t, -\eta(t,x)), \quad \text{a.e. } t \in [t_0, T), \]

\[ \psi'(t) = \psi M, \quad \text{a.e. } t \in [t_0, T), \]

where \( f = -2\mu_0 u + 2\mu_0^2 + \frac{1}{2}\mu_1^2 \). For every \( x \in S \), we know from Lemma 2.4 that \( \psi(0) \) and \( \psi(t) \) are of the same sign. Define the Lyapunov function

\[ \omega(t) = \psi(0)\psi(t) + \frac{\psi(0)}{\psi(t)} (1 + M^2(t)), \quad (t, x) \in [0, T) \times \mathbb{R}, \]

which is a positive function of \( t \in [0, T) \). From (18), it yields

\[ \omega'(t) = \psi(0)\psi'(t) - \frac{\psi(0)}{\psi^2(t)} \psi'(t)(1 + M^2(t)) + \frac{2\psi(0)}{\psi(t)} M M' \]

\[ = \frac{2\psi(0)}{\psi(t)} M f(t, -\eta(t,x)) - \frac{1}{2} \]

\[ \leq \frac{\psi(0)}{\psi(t)} (1 + M^2(t)(|f(t, -\eta(t,x))| + \frac{1}{2}) \]

\[ \leq (4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2}) \omega(t), \quad (t, x) \in [0, T) \times \mathbb{R}. \]

Using the Gronwall’s inequality, we have

\[ \omega(t) \leq \omega(0)e^{(4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2})t} \]

\[ = C_3 e^{C_2 t}, \quad t \in [0, T), \]

where \( \omega(0) = \rho^2(-x) + 1 + u_{0,x}^2(-x) \leq 1 + \| \rho_0 \|_2^2 + \| u_{0,x} \|_{L_\infty}^2 = C_3 \) and

\[ C_2 = 4\mu_0^2 + \frac{1}{2}\mu_1^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 + \frac{1}{2}. \]

Since \( \psi(t) \) and \( \psi(0) \) are of the same sign. The definition of \( \omega(t) \) implies \( |\psi(0)||M(t)| \leq \omega(t) \) and \( \psi(0)\psi(t) \leq \omega(t) \). From (21), we obtain

\[ |u_x(t, -\eta(t,x))| = |M(t)| \leq \frac{1}{|\psi(0)|} \omega(t) \]

\[ \leq \frac{1}{|\rho_0(\cdot)|} C_3 e^{C_2 t}, \quad t \in [0, T) \]

and

\[ |\rho(t, -\eta(t,x))| = |\psi(t)| \leq \frac{1}{|\psi(0)|} \omega(t) \]

\[ \leq \frac{1}{|\rho_0(\cdot)|} C_3 e^{C_2 t}, \quad t \in [0, T). \]
Now, we assume on the contrary that $T < \infty$ and the solution blows up in finite time. It follows from Theorem 3.1 that

$$\int_0^T \| u_x(t, x) \|_{L^\infty} dt = \infty. \quad (24)$$

From (22), we have

$$|u_x(t, x)| \leq \frac{1}{|\rho_0(-x)|} C_3 e^{C_2 t} < \infty, \quad (t, x) \in [0, T) \times \mathbb{R}, \quad (25)$$

which leads to a contradiction. Thus, $T = +\infty$, and the solution $z = (u, \rho)$ is global. This completes the proof of Theorem 4.1.

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