A study on logarithmically concave functions
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Abstract
For any positive real-valued function $f$ on a closed interval, we say that (1) $f$ is concave if $-f$ is convex, and (2) $f$ is logarithmically concave if $\log f$ is concave. In this paper, we present sufficient conditions for being a logarithmically concave function.

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1 Introduction

A real-valued function $f$ on a closed interval $I$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

For any positive real-valued function $f$ on a closed interval, we say that (1) $f$ is concave if $-f$ is convex, and (2) $f$ is logarithmically concave if $\log f$ is concave.

In 2012, Sulaiman [1] gave some properties concerning operations on concave functions.

For any continuous positive real-valued function $f$ on a closed interval $I$, we obtain that $f$ is logarithmically concave if and only if

$$\frac{\log f(x) + \log f(y)}{2} \leq \log f \left( \frac{x + y}{2} \right)$$

for all $x, y \in I$.

In this paper, we present sufficient conditions for being a logarithmically convex function.
2 Results

Theorem 2.1. Let \( f_1, f_2, \ldots, f_n \) be continuous positive real-valued functions on a closed interval \( I \) such that

\[
\sqrt[n]{\prod_{i=1}^{n} f_i(x) f_i(y)} \leq \min_{i=1, 2, \ldots, n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\}
\]

for all \( x, y \in I \). Then \( \prod_{i=1}^{n} f_i \) is logarithmically concave.

Proof. Let \( x, y \in I \). It follows that

\[
f_j \left( \frac{x + y}{2} \right) = \min_{i=1, 2, \ldots, n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\}
\]

for some \( j \in \{1, 2, \ldots, n\} \). Then

\[
\log \prod_{i=1}^{n} f_i(x) + \log \prod_{i=1}^{n} f_i(y) = \frac{1}{2} \log \prod_{i=1}^{n} f_i(x) f_i(y)
\]

\[
= \log \left[ \prod_{i=1}^{n} f_i(x) f_i(y) \right]
\]

\[
\leq \log \left[ \min_{i=1, 2, \ldots, n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\} \right]^n
\]

\[
= n \log \min_{i=1, 2, \ldots, n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\}
\]

\[
= n \log f_j \left( \frac{x + y}{2} \right)
\]

\[
= \sum_{i=1}^{n} \log f_j \left( \frac{x + y}{2} \right)
\]

\[
\leq \sum_{i=1}^{n} \log f_i \left( \frac{x + y}{2} \right)
\]

\[
= \log \prod_{i=1}^{n} f_i \left( \frac{x + y}{2} \right).
\]
We note that \( \prod_{i=1}^{n} f_i \) is a continuous positive real-valued function on \( I \).

Therefore \( \prod_{i=1}^{n} f_i \) is logarithmically concave.

\[ \Box \]

**Theorem 2.2.** Let \( f_1, f_2, \ldots, f_n \) and \( g \) be continuous positive real-valued functions on a closed interval \( I \) such that \( g > 1 \) and

\[
\sqrt{\frac{1}{2} \log g \left( \frac{x+y}{2} \right) g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \leq \min_{i=1,2,\ldots,n} \left\{ f_i \left( \frac{x+y}{2} \right) \right\}
\]

for all \( x, y \in I \). Then \( g^{f_1 f_2 \cdots f_n} \) is logarithmically concave.

**Proof.** Let \( x, y \in I \). It follows that

\[
f_j \left( \frac{x+y}{2} \right) = \min_{i=1,2,\ldots,n} \left\{ f_i \left( \frac{x+y}{2} \right) \right\}
\]

for some \( j \in \{1, 2, \ldots, n\} \). Then

\[
\log g \left( \frac{x+y}{2} \right) \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} = \frac{1}{2} \log g \left( \frac{x+y}{2} \right) g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)
\]

\[
\leq \left[ \min_{i=1,2,\ldots,n} \left\{ f_i \left( \frac{x+y}{2} \right) \right\} \right]^n
\]

\[
= \left[ f_j \left( \frac{x+y}{2} \right) \right]^n
\]

\[
\leq f_1 f_2 \cdots f_n \left( \frac{x+y}{2} \right).
\]

Then

\[
\sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)} \leq g^{f_1 f_2 \cdots f_n} \left( \frac{x+y}{2} \right).
\]

Then

\[
\frac{\log g^{f_1 f_2 \cdots f_n}(x) + \log g^{f_1 f_2 \cdots f_n}(y)}{2} = \frac{1}{2} \log g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)
\]

\[
= \log \sqrt{g^{f_1 f_2 \cdots f_n}(x) g^{f_1 f_2 \cdots f_n}(y)}
\]

\[
\leq \log g^{f_1 f_2 \cdots f_n} \left( \frac{x+y}{2} \right).
\]
We note that \( g_{f_1 f_2 \ldots f_n} \) is a continuous positive real-valued function on \( I \). Therefore \( g_{f_1 f_2 \ldots f_n} \) is logarithmically concave.

A real-valued function \( f \) on a closed interval \( I \) is said to be quasi-concave if

\[
 f(tx + (1 - t)y) \geq \min \{ f(x), f(y) \}
\]

for all \( x, y \in I \) and \( t \in [0, 1] \).

Moreover, any positive logarithmically concave function is also quasi-concave.

**Corollary 2.3.** Let \( f_1, f_2, \ldots, f_n \) be continuous positive real-valued functions on a closed interval \( I \) such that

\[
 2^n \prod_{i=1}^{n} f_i(x) f_i(y) \leq \min_{i=1,2,\ldots,n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\}
\]

for all \( x, y \in I \). Then \( \prod_{i=1}^{n} f_i \) is quasi-concave.

**Corollary 2.4.** Let \( f_1, f_2, \ldots, f_n \) and \( g \) be continuous positive real-valued functions on a closed interval \( I \) such that \( g > 1 \) and

\[
 \sqrt[n]{\frac{1}{2} \log_{g(\frac{x+y}{2})} g_{f_1 f_2 \ldots f_n}(x) g_{f_1 f_2 \ldots f_n}(y)} \leq \min_{i=1,2,\ldots,n} \left\{ f_i \left( \frac{x + y}{2} \right) \right\}
\]

for all \( x, y \in I \). Then \( g_{f_1 f_2 \ldots f_n} \) is quasi-concave.

**References**


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