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A note on the asymptotics of the modified Bessel functions on the Stokes lines

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Abstract

We employ the exponentially improved asymptotic expansions of the confluent hypergeometric functions on the Stokes lines discussed by the author [Appl. Math. Sci. **7** (2013) 6601–6609] to give the analogous expansions of the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ for large z and finite ν on arg $z = \pm\pi$ (and, in the case of $I_\nu(z)$, also on arg $z = 0$). Numerical results are presented to illustrate the accuracy of these expansions.

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1. Introduction The interest in exponentially precise asymptotics during the past three decades has shown that retention of exponentially small terms, previously neglected in asymptotics, can be essential for a high-precision description. For a discussion of recent developments in this area see [2, Section 2.11] and [3, Chapter 6]. An example illustrating the advantage of retaining exponentially small terms in the asymptotic expansion of a certain integral is given in [2, p. 66]. Although such terms are negligible in the Poincaré sense, their inclusion can significantly improve the numerical accuracy.

The modified Bessel function of the first kind $I_\nu(z)$ is defined by

$$I_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{2}z)^{2k}}{k!\Gamma(k+\nu+1)}. \tag{1.1}$$

When ν takes on half-integer values, $I_\nu(z)$ can be expressed in terms of the hyperbolic functions; we exclude this special case from our asymptotic considerations. The behaviour of $I_\nu(z)$ for large z and fixed ν is exponentially large throughout the sector $|\arg z| \le \pi$, except on the imaginary axis $z = \pm ix$, $x > 0$, where it is oscillatory (and is equal to $e^{\pm\pi\nu i/2}J_\nu(x)$, where $J_\nu(z)$ is the Bessel function of the first kind). The well-known asymptotic expansion of $I_\nu(z)$ for $|z| \to \infty$ and finite ν is given by [6, p. 203], [2, (10.40.5)]

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{(-)^k a_k(\nu)}{z^k} + \frac{ie^{-z+\pi\nu i}}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} \tag{1.2}$$

valid in the sector $-\frac{1}{2}\pi + \epsilon \le \arg z \le \frac{3}{2}\pi - \epsilon$, where throughout ϵ denotes an arbitrary small constant. An analogous expansion with the sign of i reversed holds in the conjugate sector $-\frac{3}{2}\pi + \epsilon \le \arg z \le \frac{1}{2}\pi - \epsilon$. The coefficients $a_k(\nu)$ are defined by[1]

$$a_k(\nu) = \frac{(-)^k}{2^k k!}\,(\tfrac{1}{2}+\nu)_k(\tfrac{1}{2}-\nu)_k \qquad (k \ge 0), \tag{1.3}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol.

The first series in (1.2) is dominant as $|z| \to \infty$ in the right-half plane, with the second series being subdominant and maximally subdominant on $\arg z = 0$. This situation is reversed in the left-half plane; the second series becomes dominant and assumes maximum dominance over the first series on the Stokes lines $\arg z = \pm\pi$. On these rays the subdominant first series undergoes a Stokes phenomenon and begins to switch off (in the sense of increasing $|\arg z|$). An analogous process occurs on the other Stokes line $\arg z = 0$, where the subdominant second series in (1.2) undergoes a similar transition. A correct interpretation of the expansion (1.2) would require the dominant series to be optimally truncated at, or near, its least term in magnitude (corresponding to the truncation index $k \simeq 2|z|$) in order to produce a remainder that is comparable to the subdominant contribution.

However, these Stokes transitions are not fully accounted for in the expansion (1.2). To see this, we put $z = xe^{\pi i}$, $x > 0$, so that we have from (1.2)

$$e^{-\pi\nu i} I_\nu(xe^{\pi i}) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{(-)^k a_k(\nu)}{x^k} - \frac{ie^{-x-\pi\nu i}}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{x^k} \tag{1.4}$$

as $x \to +\infty$. For real ν, the dominant contribution in (1.4) is real, as it must be since from (1.1) we have

$$I_\nu(xe^{\pm\pi i}) = e^{\pm\pi\nu i} I_\nu(x) \tag{1.5}$$

with the consequence that $e^{-\pi\nu i} I_\nu(xe^{\pi i})$ is real in this case. However, the expansion (1.4) predicts a complex-valued exponentially small contribution

[1] This representation of the coefficients is equivalent to the familiar form $a_k(\nu) = (4\nu^2 - 1^2)(4\nu^2 - 3^2)\ldots(4\nu^2 - (2k-1)^2)/(2^{3k}k!)$ for $k \ge 1$.

(when ν is not a half-integer). The same reasoning applies to $I_\nu(x)$ on the Stokes line $x \in [0, \infty)$ since, from (1.5), its asymptotic expansion for $x \to +\infty$ is also given by the right-hand side of (1.4).

The modified Bessel function of the second kind $K_\nu(z)$ has the expansions

$$K_\nu(z) \sim \begin{cases} \sqrt{\dfrac{\pi}{2z}}\, e^{-z} \displaystyle\sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} & (|\arg z| \leq \tfrac{3}{2}\pi - \epsilon) \\ \sqrt{\dfrac{\pi}{2z}}\, e^{-z} \displaystyle\sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} + 2i\cos\pi\nu \sqrt{\dfrac{\pi}{2z}}\, e^{z} \sum_{k=0}^{\infty} \frac{(-)^k a_k(\nu)}{z^k} & \\ \qquad (\tfrac{1}{2}\pi + \epsilon \leq \arg z \leq \tfrac{5}{2}\pi - \epsilon). & \end{cases}$$

Consequently, on the Stokes line $\arg z = \pi$, we obtain with $z = xe^{\pi i}$, $x > 0$

$$K_\nu(xe^{\pi i}) \sim -i\sqrt{\frac{\pi}{2x}}\, e^{x} \sum_{k=0}^{\infty} \frac{(-)^k a_k(\nu)}{x^k} + 2\cos\pi\nu \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{x^k} \tag{1.6}$$

as $x \to +\infty$. A conjugate expansion holds on $\arg z = -\pi$. However, the expansion (1.6) will not yield the correct form for the exponentially small contribution for large x, since on $\arg z = \pi$ the appearance of the subdominant second series in (1.6) is only halfway through its Stokes transition.

In this note we employ the exponentially improved asymptotic expansions of the confluent hypergeometric functions on the Stokes lines discussed in [4] to deal with the analogous expansions of the modified Bessel functions. We obtain the exponentially small contribution associated with $I_\nu(z)$ on the Stokes lines $\arg z = 0$ and $\arg z = \pi$, and that associated with $K_\nu(z)$ on the Stokes line $\arg z = \pi$, that take into account the incomplete nature of the Stokes transitions on these rays.

2. A summary of the asymptotic expansions of the Kummer functions We present in this section a summary of the exponentially improved expansion of the Kummer functions ${}_1F_1(a; b; -x)$ and $U(a, b, xe^{\pm\pi i})$ for $x \to +\infty$ with fixed parameters a and b.

The function $U(a, b, z)$ has, in general, a branch point at $z = 0$ with the z-plane cut along $(-\infty, 0]$ and possesses the large-$|z|$ expansion

$$U(a, b, z) \sim z^{-a} \sum_{k=0}^{\infty} \frac{(-)^k (a)_k (1 + a - b)_k}{k!\, z^k} \qquad (|\arg z| \leq \tfrac{3}{2}\pi - \epsilon),$$

which is algebraic in character in the stated sector. However, on the Stokes lines $\arg z = \pm\pi$, an exponentially small expansion switches on, so that in the sectors $\pi \leq |\arg z| \leq \frac{3}{2}\pi - \epsilon$ we have a compound expansion with a subdominant exponential contribution. This latter contribution becomes dominant beyond $|\arg z| = \frac{3}{2}\pi$.

In the following we let M, N, m denote positive integers and define the parameters

$$\vartheta := a - b, \qquad \mu \equiv \mu(m) := 2a - b + m. \tag{2.1}$$

The exponentially improved expansion of $U(a,b,z)$ is given by [2, p. 329]

$$U(a,b,z) = z^{-a} \sum_{k=0}^{m-1} \frac{(-)^k (a)_k (1+a-b)_k}{k!\, z^k}$$

$$+ \frac{2\pi i e^{-\pi i(a+\vartheta)}}{\Gamma(a)\Gamma(1+a-b)}\, z^{\vartheta} e^{z} \left\{ \sum_{j=0}^{M-1} A_j z^{-j}\, T_{\nu-j}(z) + R_{M,m}(z) \right\}, \tag{2.2}$$

where

$$A_j = \frac{(1-a)_j (1-b)_j}{j!} \qquad (j \geq 0) \tag{2.3}$$

and $T_\mu(z)$ denotes the so-called *terminant function* defined as a multiple of the incomplete gamma function $\Gamma(a,z)$ by

$$T_\mu(z) := \frac{e^{\pi i \mu}\Gamma(\mu)}{2\pi i}\, \Gamma(1-\mu, z).$$

In (2.2), m is an arbitrary positive integer but will be chosen to be the optimal truncation index m_o of the algebraic expansion corresponding to truncation at, or near, the least term in magnitude. This is easily verified to be

$$m_o \simeq |z| - \Re(2a - b), \tag{2.4}$$

so that $m_o \to \infty$ as $|z| \to +\infty$. When $m - |z|$ is bounded, the remainder term in (2.2) satisfies $R_{M,m}(z) = O(e^{-|z|-z} z^{-M})$ as $|z| \to \infty$ in the sector $|\arg z| \leq \pi$.

Since the truncation index m is chosen to be optimal, the index μ appearing in (2.1) satisfies $\mu \sim |z|$ as $|z| \to +\infty$. The asymptotic expansion of $T_\mu(z)$ for large $|\mu|$ and $|z|$, when $|\mu| \sim |z|$, has been discussed in detail by Olver in [1]. By expressing $T_\mu(z)$ in terms of a Laplace integral, which is associated with a saddle point and a simple pole becoming coincident on $\arg z = \pi$, Olver [1, §5] established that, for $z = xe^{\pi i}$, $x > 0$,

$$T_{\mu-j}(xe^{\pi i}) = \frac{1}{2} - \frac{i}{\sqrt{2\pi x}} \left\{ \sum_{k=0}^{N-1} (\tfrac{1}{2})_k G_{2k,j} (\tfrac{1}{2}x)^{-k} + O(x^{-N}) \right\} \qquad (x \to +\infty), \tag{2.5}$$

where the coefficients $G_{k,j}$ result from the expansion

$$\frac{\tau^{\gamma_j - 1}}{1-\tau} \frac{d\tau}{dw} = -\frac{1}{w} + \sum_{k=0}^{\infty} G_{k,j} w^k, \qquad \tfrac{1}{2}w^2 = \tau - \log \tau - 1.$$

The branch of $w(\tau)$ is chosen such that $w \sim \tau - 1$ as $\tau \to 1$ and the parameter γ_j is specified by

$$\gamma_j = \mu - x - j \qquad (0 \le j \le N-1) \tag{2.6}$$

with $|\gamma_j|$ bounded. Upon reversion of the w-τ mapping to yield

$$\tau = 1 + w + \tfrac{1}{3}w^2 + \tfrac{1}{36}w^3 - \tfrac{1}{270}w^4 + \tfrac{1}{4320}w^5 + \cdots\ ,$$

it is found with the help of *Mathematica* that the first five even-order coefficients $G_{2k,j} \equiv 6^{-2k}\hat{G}_{2k,j}$ are[2]

$$\begin{aligned}
\hat{G}_{0,j} &= \tfrac{2}{3} - \gamma_j, \qquad \hat{G}_{2,j} = \tfrac{1}{15}(46 - 225\gamma_j + 270\gamma_j^2 - 90\gamma_j^3),\\
\hat{G}_{4,j} &= \tfrac{1}{70}(230 - 3969\gamma_j + 11340\gamma_j^2 - 11760\gamma_j^3 + 5040\gamma_j^4 - 756\gamma_j^5),\\
\hat{G}_{6,j} &= \tfrac{1}{350}(-3626 - 17781\gamma_j + 183330\gamma_j^2 - 397530\gamma_j^3 + 370440\gamma_j^4 - 170100\gamma_j^5\\
&\qquad +37800\gamma_j^6 - 3240\gamma_j^7),\\
\hat{G}_{8,j} &= \tfrac{1}{231000}(-4032746 + 43924815\gamma_j + 88280280\gamma_j^2 - 743046480\gamma_j^3\\
&\qquad +1353607200\gamma_j^4 - 1160830440\gamma_j^5 + 541870560\gamma_j^6 - 141134400\gamma_j^7\\
&\qquad +19245600\gamma_j^8 - 1069200\gamma_j^9).
\end{aligned}$$

Substitution of (2.5) into (2.2) (where for convenience we put $M = N$) and introduction of the coefficients B_j defined by

$$B_j = \sum_{k=0}^{j} (-2)^k (\tfrac{1}{2})_k\, A_{j-k}\, G_{2k,j-k}, \tag{2.7}$$

then yields the expansion:

Theorem 1. [*4*, (3.2)] *We have the expansion*

$$U(a,b,xe^{\pm\pi i}) - (xe^{\pm\pi i})^{-a} \sum_{k=0}^{m_o-1} \frac{(a)_k(1+a-b)_k}{k!\, x^k}$$

$$= \pm\frac{2\pi i\, e^{\mp\pi i a}\, x^{a-b} e^{-x}}{\Gamma(a)\Gamma(1+a-b)} \left\{ \frac{1}{2}\sum_{j=0}^{M-1} (-)^j A_j x^{-j} \mp \frac{i}{\sqrt{2\pi x}} \sum_{j=0}^{M-1} (-)^j B_j x^{-j} + O(x^{-M}) \right\} \tag{2.8}$$

as $x \to +\infty$*, provided* $a,\ 1+a-b \ne 0, -1, -2, \ldots$*. The integer* m_o *is the optimal truncation index of the algebraic expansion satisfying* $m_o \sim x$*,* M *is a positive integer and the coefficients* A_j *and* B_j *are defined in (2.3) and (2.7), respectively.*

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[2] There was a misprint in the first term in $\hat{G}_{6,j}$ in [4], which appeared as -3226 instead of -3626. This was pointed out by T. Pudlik [5]. The correct value was used in the numerical calculations described in [4].

From the relation connecting the first Kummer function ${}_1F_1(a;b;-x)$ to $U(a,b,xe^{\pm\pi i})$ [2, p. 323] we obtain the expansion:

Theorem 2. [4, (2.11)] *When $\vartheta = a-b$ is non-integer we have the expansion*

$$\frac{\Gamma(a)}{\Gamma(b)}\,{}_1F_1(a;b;-x)-\frac{x^{-a}\Gamma(a)}{\Gamma(b-a)}\sum_{k=0}^{m_o-1}\frac{(a)_k(1+a-b)_k}{k!\,x^k}$$

$$=x^{a-b}e^{-x}\biggl\{\cos\pi\vartheta\sum_{j=0}^{M-1}(-)^jA_jx^{-j}-\frac{2\sin\pi\vartheta}{\sqrt{2\pi x}}\sum_{j=0}^{M-1}(-)^jB_jx^{-j}+O(x^{-M})\biggr\} \tag{2.9}$$

as $x\to+\infty$, where m_o is the optimal truncation index of the algebraic expansion satisfying $m_o\sim x$, M is a positive integer and the coefficients A_j and B_j are defined in (2.3) and (2.7), respectively.

When $\vartheta = n$, $n = 0, 1, 2, \ldots$, the algebraic expansion in (2.9) vanishes and the coefficients A_j vanish for $j > n$. In this case the function ${}_1F_1(a;b;z)$ is a polynomial in z of degree n. When $\vartheta = -n$, $n = 1, 2, \ldots$, the algebraic expansions in (2.8) and (2.9) consist of n terms (and so cannot be optimally truncated); see [4] for details. In both cases the second exponentially small series in (2.9) vanishes.

3. The expansions on the Stokes lines From the identity expressing $I_\nu(z)$ in terms of the confluent hypergeometric function [2, (10.39.5)] we have, with $z = xe^{\pi i}$, $x > 0$

$$e^{-\pi\nu i}I_\nu(xe^{\pi i})=\frac{(\frac{1}{2}x)^\nu e^x}{\Gamma(\nu+1)}\,{}_1F_1(\nu+\tfrac{1}{2};2\nu+1;-2x).$$

With the parameter $\vartheta = -\nu-\frac{1}{2}$, we obtain from Theorem 2 the expansion

$$e^{-\pi\nu i}I_\nu(xe^{\pi i})=\frac{(\frac{1}{2}x)^\nu e^x}{\Gamma(\nu+1)}\,\frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)}\biggl\{(2x)^{-\nu-\frac{1}{2}}\sum_{k=0}^{m_o-1}\frac{(\frac{1}{2}+\nu)_k(\frac{1}{2}-\nu)_k}{k!(2x)^k}$$

$$+(2x)^{-\nu-\frac{1}{2}}e^{-2x}\biggl\{\cos\pi\vartheta\sum_{j=0}^{M-1}\frac{(-)^jA_j}{(2x)^j}-\frac{2\sin\pi\vartheta}{\sqrt{4\pi x}}\sum_{j=0}^{M-1}\frac{(-)^jB_j}{(2x)^j}+O(x^{-M})\biggr\}\biggr\}.$$

Use of the duplication formula for the gamma function

$$\Gamma(2z)=2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

and the fact that the coefficients $a_j(\nu) = (-2)^{-j}A_j$, where A_j are defined in (2.3) with $a = \nu+\frac{1}{2}$, $b = 2\nu+1$, then produces

Theorem 3. *Let $\vartheta = -\nu - \frac{1}{2}$ and M be a positive integer. Then we have the expansion*

$$e^{-\pi\nu i} I_\nu(xe^{\pi i}) = \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{m_o-1} \frac{(-)^k a_k(\nu)}{x^k}$$

$$+\frac{e^{-x}}{\sqrt{2\pi x}} \biggl\{ \cos\pi\vartheta \sum_{k=0}^{M-1} \frac{a_k(\nu)}{x^k} - \frac{\sin\pi\vartheta}{\sqrt{\pi x}} \sum_{j=0}^{M-1} \frac{(-)^j B_j}{(2x)^j} + O(x^{-M}) \biggr\} \quad (3.1)$$

as $x \to +\infty$, where m_o is the optimal truncation index for the dominant series satisfying $m_o \simeq 2x$. The coefficients $a_k(\nu)$ and B_j are defined in (1.3) and (2.7), respectively.

We remark that the right-hand side of (3.1) also gives the expansion of $e^{\pi\nu i} I_\nu(xe^{-\pi i})$ as well as that of $I_\nu(x)$ as $x \to +\infty$ since, by (1.5), $I_\nu(x) = e^{\mp\pi\nu i} I_\nu(xe^{\pm\pi i})$.

The modified Bessel function $K_\nu(z)$ is given by

$$K_\nu(z) = (2z)^\nu \sqrt{\pi} e^{-z}\, U(\nu + \tfrac{1}{2}, 2\nu + 1, 2z).$$

From Theorem 1 we then find[3] with $z = xe^{\pm\pi i}$, $x > 0$ that

$$K_\nu(xe^{\pm\pi i}) = (2xe^{\pm\pi i})^\nu \sqrt{\pi} e^x \Biggl\{ (2xe^{\pm\pi i})^{-\nu-\frac{1}{2}} \sum_{k=0}^{m_o-1} \frac{(\frac{1}{2}+\nu)_k (\frac{1}{2}-\nu)_k}{k!(2x)^k}$$

$$\pm \frac{2\pi i\,(2xe^{\pm\pi i})^{-\nu-\frac{1}{2}} e^{-2x}}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \biggl\{ \frac{1}{2} \sum_{j=0}^{M-1} \frac{(-)^j A_j}{(2x)^j} \mp \frac{i}{2\sqrt{\pi x}} \sum_{j=0}^{M-1} \frac{(-)^j B_j}{(2x)^j} + O(x^{-M}) \biggr\} \Biggr\}.$$

After some straightforward rearrangement, this produces

Theorem 4. *Let M be a positive integer. Then we have the expansions*

$$K_\nu(xe^{\pm\pi i}) = \mp i \sqrt{\frac{\pi}{2x}}\, e^x \sum_{k=0}^{m_o-1} \frac{(-)^k a_k(\nu)}{x^k}$$

$$+2\cos\pi\nu \sqrt{\frac{\pi}{2x}} e^{-x} \biggl\{ \frac{1}{2} \sum_{k=0}^{M-1} \frac{a_k(\nu)}{x^k} \mp \frac{i}{2\sqrt{\pi x}} \sum_{j=0}^{M-1} \frac{(-)^j B_j}{(2x)^j} + O(x^{-M}) \biggr\} \quad (3.2)$$

as $x \to +\infty$, where m_o is the optimal truncation index for the dominant series satisfying $m_o \simeq 2x$. The coefficients $a_k(\nu)$ and B_j are defined in (1.3) and (2.7), respectively.

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[3]This result can also be obtained from (10.40.2) and (10.40.13) in [2] combined with the expansion (2.5).

Comparison of (3.2) with (1.6) reveals the by-now familiar fact that the value of the Stokes multiplier of $K_\nu(z)$ on $\arg z = \pm\pi$ (given by the expression in curly braces in (3.2)) is $\frac{1}{2}$ to leading order.

4. Numerical calculations In numerical calculations, we set the optimal truncation index $m_o = 2x + \alpha$, where $|\alpha| \leq \frac{1}{2}$; when $2x$ is an integer then $\alpha = 0$. Then from (2.1) and (2.6) we have

$$\gamma_j = m_o - 2x - j = \alpha - j.$$

The coefficients B_j can be computed from (2.7) and are real when ν is real. We subtract off the dominant, exponentially large series in (3.1) by defining

$$F_\nu(x) := e^{-\pi\nu i} I_\nu(xe^{\pi i}) - \frac{e^x}{\sqrt{2\pi x}} \sum_{k=0}^{m_o-1} \frac{(-)^k a_k(\nu)}{x^k}\ .$$

The exponentially small expansion is given by

$$S_I(M;x) := \frac{e^{-x}}{\sqrt{2\pi x}} \left\{ \cos \pi\vartheta \sum_{k=0}^{M-1} \frac{a_k(\nu)}{x^k} - \frac{\sin \pi\vartheta}{\sqrt{\pi x}} \sum_{j=0}^{M-1} \frac{(-)^j B_j}{(2x)^j} \right\},$$

which is seen to be real when ν is real. In Table 1 we show[4] the values of $S_I(M;x)$ as a function of the truncation index M for different values of x and compare these with the value of $F_\nu(x)$.

Table 1: The values of $F_\nu(x)$ and $S_I(M;x)$ for $e^{-\pi\nu i} I_\nu(xe^{\pi i})$ for $x = 25$ and different truncation index M when $\nu = 1/4$.

M	$S_I(M;10)$	$S_I(M;15.4)$	$S_I(M;20)$
1	−3.568247262(−06)	−1.163196884(−08)	−1.190793339(−10)
2	−3.538491386(−06)	−1.151737379(−08)	−1.185631040(−10)
3	−3.539961940(−06)	−1.151900437(−08)	−1.185763303(−10)
4	−3.539827969(−06)	−1.151884096(−08)	−1.185756985(−10)
5	−3.539846361(−06)	−1.151885440(−08)	−1.185757438(−10)
6	−3.539842998(−06)	−1.151885272(−08)	−1.185757394(−10)
7	−3.539843764(−06)	−1.151885298(−08)	−1.185757400(−10)
$F_\nu(x)$	−3.539843604(−06)	−1.151885294(−08)	−1.185757399(−10)

[4] In Tables 1 and 2 we write the values as $x(y)$ instead of $x \times 10^y$.

Similarly, we define

$$G_\nu(x) := K_\nu(xe^{\pi i}) + i\sqrt{\frac{\pi}{2x}}\, e^x \sum_{k=0}^{m_o-1} \frac{(-)^k a_k(\nu)}{x^k}$$

and

$$S_K(M;x) := 2\cos\pi\nu\,\sqrt{\frac{\pi}{2x}} e^{-x}\left\{\frac{1}{2}\sum_{k=0}^{M-1}\frac{a_k(\nu)}{x^k} - \frac{i}{2\sqrt{\pi x}}\sum_{j=0}^{M-1}\frac{(-)^j B_j}{(2x)^j}\right\}.$$

Table 2 shows an example of the values of $S_K(M;x)$ for different truncation index M compared with the value of $G_\nu(x)$. It can be seen in both cases that the computed values of $F_\nu(x)$ and $G_\nu(x)$ agree well their corresponding exponentially small expansions.

Table 2: The values of $G_\nu(x)$ and $S_K(M;x)$ for $K_\nu(xe^{\pi i})$ for different truncation index M when $x = 25$ and $\nu = 1/4$.

M	$S_K(M;25)$
1	$2.461573958(-12) - 1.851725849i(-13)$
2	$2.452343056(-12) - 1.839098107i(-13)$
3	$2.452544982(-12) - 1.839470730i(-13)$
4	$2.452536653(-12) - 1.839451010i(-13)$
5	$2.452537160(-12) - 1.839452410i(-13)$
6	$2.452537119(-12) - 1.839452283i(-13)$
7	$2.452537123(-12) - 1.839452297i(-13)$
$G_\nu(x)$	$2.452537123(-12) - 1.839452296i(-13)$

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